Honors Introduction to Ordinary and Partial Differential Equations: Penn State Math 251H Lecture Notes

Version 1.0

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Using These Notes

Welcome to the eleventh set of course lecture notes I've written. This is a set of lecture notes for Math 251H, Honors Ordinary and Partial Differential Equations. The first question you should be asking is, "Why would you write lecture notes for an introductory course when there are dozens if not hundreds of books on this subject?" To be honest, I hadn't planned on writing these notes and I wouldn't have if I'd been teaching the coordinated non-honors course. There just isn't enough freedom to adjust the material, which largely follows Boyce and DiPrima's book [**BDM21**]. The honors section is an entirely different course whose delivery is largely up to the instructor as long as he/she/they cover the material that would be covered in the non-honors sections. So, I wrote these notes.

The lecture notes are loosely based on a number of books, all of which are fine but none of which do I love. Obviously, since Penn State uses Boyce and DiPrima [**BDM21**] for almost all sections of Math 251, the structure of the notes is somewhat based on the order of that book. Former instructors of the honors section used Nagle, Saff and Snider [**NSS11**] which is a good book, but didn't have everything I wanted. It did have a good treatment of the Energy Theorem, which I made sure to include. I also used Borelli and Coleman's [**BC98**] book, which is out of print. I really liked this book and a lot of the spirit of the notes is drawn from there. In addition to these more general references, I also took inspiration from Adkins and Davidson [**AD12**], which I think is more geared toward a junior or senior level treatment. Examples are also taken from Arnold [**Arn92**] (which is definitely not geared toward undergraduates) and, of course, Strogatz [**Str18**]. The partial differential equations material is taken from my notes on the subject, which are largely derived from [**Asm16, Hab03, Olv14, Log14**], but we only touch on the rudimentary aspects of the subject.

I used to advise people to get one of the books I reference and you probably should, but it's not required. These notes most likely have typos left (but nothing egregious). There is a rather amusing story of a typo we found in class that resulted from a Mathematica gaff on my part – but that's been fixed. That said, the notes do have some idiosyncrasies that are unique to me. I spend very little time on transient behavior in damped oscillators – mainly because I don't find it terribly interesting. However, following [**BC98**] we do talk about beat frequency. The Laplace transform is used mainly to get at Green's functions, on which we briefly touch and I unabashedly use the fact that that Laplace transform variable (usually s) is imaginary. In systems, I eschew the usual discussion and go straight to the solution as a matrix exponent. The information on matrix diagonalization is taken from my Linear Algebra notes. We discuss but don't prove the Jordan decomposition to explain why $te^{\lambda}t$ shows up in solutions with degenerate eigenspaces because I cannot abide the "let's guess a solution..." approach used in [**BDM21**]. Most classes never get to the explanation of why that guess was right, so it leaves one feeling rather empty – like Math is some sort of series of lucky guesses and tricks, which I hate. Finally I tried writing these in "lesson" and "module" format rather than as chapters. I ended up not liking it much, since I taught 59 lectures but there are 44 ‘lessons’, which means some lessons took more than one lecture. Perhaps one day I’ll reformat the notes into standard chapters. Given all those caveats, I hope you enjoy using these notes.

Module 1

Introduction to Differential Equations

LESSON 1

1. Some Preliminaries

Remark 1.1. These lectures assume familiarity with all the material in the first two semesters of calculus. This includes explicit and implicit differentiation methods and methods of integration. However, since vector calculus is not a prerequisite, we will very quickly review functions of several variables, partial derivatives and implicit functions.

Remark 1.2. The expression $D \subseteq \mathbb{R}^n$ means that D is a set contained in or equal to $\mathbb{R}^n$, the n-dimensional Euclidean space. Here the set $\mathbb{R}$ is the set of all real numbers, not including $\pm\infty$. We don't need to know what a dimension really is for now; it suffices to think of it as the number of axes when you "draw a picture."

Definition 1.3 (Function of Several Variables)**.** A function $u : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function of n variables and is usually written $u(x_1, x_2, \ldots, x_n)$ with domain D. When $n = 2$, we may write $u(x, y)$ (for simplicity) and when $n = 3$ we may write $u(x, y, z)$.

Example 1.4. The following are some functions of several variables:

$$u(x, y) = x^2 + y^2 \tag{1.1}$$

$$v(x, y, z, t) = x^2 + y^2 + z^2 - c^2t^2. \tag{1.2}$$

The first function is simply a generalization of a parabola. Plotting $z = u(x, y)$ yields a two-dimensional surface embedded in $\mathbb{R}^3$ called a paraboloid, see Fig. 1.1(left). The second

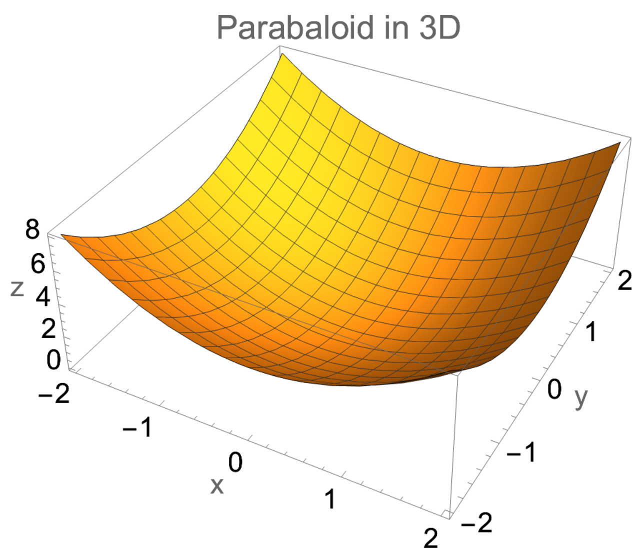


FIGURE 1.1. A paraboloid in $\mathbb{R}^3$ defined by the equation $z = u(x, y)$.

function is used to define Minkowski spacetime in special relativity.

Definition 1.5 (Partial Derivative)**.** Let $u : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function of several variables. The partial derivative with respect to x_i (if it exists) is:

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{u(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - u(x_1, \dots, x_n)}{h}. \tag{1.3}$$

Remark 1.6. Definition 1.5 really just says that if $u(x_1, \dots, x_n)$ is function of many variables, then a partial derivative with respect to x_i treats all other variables as constants.

Example 1.7. Consider the function $u(x, y) = x \log(y)$. Then:

$$\frac{\partial u}{\partial x} = \log(y) \tag{1.4}$$

$$\frac{\partial u}{\partial y} = \frac{x}{y}. \tag{1.5}$$

Definition 1.8 (Implicit Function)**.** Suppose $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function of several variables, then if we have $f(x_1, \dots, x_n) = 0$, then we have x_n defined as an implicit function of $x_1, \dots, x_{n-1}$.

Remark 1.9. The expression $x \in X$ means x is in the set X.

Example 1.10. Let $a, b \in \mathbb{R}$. The function:

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \tag{1.6}$$

can be used to define ellipses with:

$$f(x, y) = 0 \iff \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1.7}$$

That is, you will recall from Algebra 2 that the curve:

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$$

defines an ellipse with center at $(0, 0)$.

Remark 1.11. In Definition 1.8 we could easily replace x_n with any of the x_i's. We used x_n for convenience. We also note that the existence of an explicit function (e.g.) $y = g(x)$ given $f(x, y) = C$ is not guaranteed except under certain conditions given by the *implicit function theorem*. For completeness, we state the implicit function theorem for $\mathbb{R}^2$.

THEOREM 1.12 (Implicit Function Theorem (2D)). *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a function. Consider the set:*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : u(x, y) = 0\} \tag{1.8}$$

Suppose $(x_0, y_0) \in \Gamma$. If

$$\left. \frac{\partial u}{\partial y} \right|_{x = x_0, y = y_0} \neq 0,$$

then there is a region $N \subset \mathbb{R}$ with $(x_0, y_0) \in N$ and a continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ so that for $(x, y) \in N$ we have $y = f(x)$.

Example 1.13. You are already familiar with this to some extent. Assuming $y > 0$ and $x \in [-a, a]$ we know that the ellipse can be modeled by the equation:

$$y = \sqrt{b^2 \left(1 - \frac{x^2}{a^2}\right)}.$$

Thus our neighborhood N is about $x_0 = 0$, $y_0 = b$ and is the rectangle $[-a, a] \times [0, b]$.

2. Ordinary Differential Equations

Definition 1.14. An *ordinary differential equation* (ODE) is an equation involving an *unknown* function of one variable $y(x)$ and any number of its derivatives. We generally assume that x is restricted to a known interval I.

Remark 1.15 (Notation)**.** Let $y(x)$ be a function of the independent variable x. Then we can write the first derivative of y as:

$$\frac{dy}{dx} = y'(x). \tag{1.9}$$

The n^{th} derivative can be written:

$$\frac{d^n y}{dx^n} = y^{(n)}(x), \tag{1.10}$$

with second derivatives usually written $y''(x)$ and third derivatives written $y'''(x)$. The exception to this notation is when the independent variable is time (or time like) in which case we may have a function $x(t)$. Then the first derivative can (but doesn't have to be) written as:

$$\frac{dx}{dt} = \dot{x} \tag{1.11}$$

$$\frac{d^2x}{dt^2} = \ddot{x} \tag{1.12}$$

$$\vdots \tag{1.13}$$

$$\frac{d^n x}{dt^n} = x^{(n)}(t). \tag{1.14}$$

The "dot notation" is a hold-over from Newton's Fluxion Notation, while the rest of the notation is due to Leibniz.

Remark 1.16 (Notation Remark)**.** In these notes, notation is intentionally mixed so that you become used to all the varieties you might encounter in future classes. That is, we intentionally switch dependent variable names, independent variable names, derivative markings etc. However, notation will be consistent within any single remark, derivation, theorem etc.

Remark 1.17. In general, we can write an ordinary differential equation as:

$$F(y, y', \dots, y^{(n)}, x) = 0, \tag{1.15}$$

where F represents a function acting on the unknown function $y(x)$, its derivatives and its independent variable $x \in I$.

Definition 1.18 (Order)**.** Consider an ordinary differential equation $F(y, y', \dots, y^{(n)}, x) = g(x)$. The *order* of the ODE is n, the degree of the highest derivative appearing in the equation.

Example 1.19 (Galilean Gravity)**.** The following is a simple second order ODE:

$$\ddot{y} + g = 0, \tag{1.16}$$

where $g \in \mathbb{R}$ is a constant and $g > 0$. Notice in this equation we have suppressed the independent variable x. This is standard unless the independent variable appears explicitly in a function. This is a model of Galilean gravity; that is, gravity near the surface of the earth without air resistance. Here, $g \approx 9.8m/s$ is the gravitational constant.

Example 1.20 (Exponential Growth/Decay)**.** The following is a simple first order ODE:

$$y' - \alpha y = 0, \tag{1.17}$$

where $\alpha \in \mathbb{R}$ is a constant. This differential equation models exponential growth (or decay). We will study it in much greater detail shortly.

Example 1.21 (Beam Equation)**.** The following fourth order ODE is called the *beam equation*:

$$\frac{d^4w}{dx^4} = q(x), \tag{1.18}$$

where $q(x)$ is a function that is specific to the beam being described. The beam equation describes the deflection of a solid but flexible beam as a result of force. It is used in structural engineering [**Asm16**].

Definition 1.22 (Explicit Solution)**.** Suppose that $F(y, y', \dots, y^{(n)}, x) = 0$ with $x \in I$ is an ODE. An explicit solution is a function $y = \varphi(x)$ that satisfies the equation for all $x \in I$. That is, when $\varphi(x)$ is substituted for y, then the equation defining the ODE is true for all $x \in I$.

Example 1.23. Consider the model of Galilean gravity. The function $y(t) = C_0 + C_1 t - \frac{1}{2}gt^2$ is an explicit solution where $C_0, C_1 \in \mathbb{R}$. To see this, simply compute:

$$\ddot{y} = -g. \tag{1.19}$$

Thus we see that $\ddot{y} + g = -g + g = 0$. This method of verification can be applied anytime you are given an explicit solution.

Definition 1.24 (Implicit Solution)**.** Suppose that $G(x, y) = 0$ is an implicit definition of y in terms of x. Then $G(x, y)$ is an implicit solution of the ODE $F(y, y', \dots, y^{(n)}, x) = 0$ with $x \in I$ if it defines one or more explicit solutions of the ODE.

Example 1.25. Consider the ODE:

$$yy' + x = 0, \tag{1.20}$$

with $x, y \in (0, +\infty)$. We claim that $G(x, y) = x^2 + y^2 - C^2 = 0$ is a solution to the ODE for any $C \in \mathbb{R}$. To see this, note if $x^2 + y(x)^2 - C^2 = 0$, then using implicit differentiation yields:

$$x + yy' = 0,$$

which is the ODE in question. Notice we assumed that $G(x, y)$ defined a function $y = g(x)$. We need the implicit function theorem to make this rigorous – which we will not do because $G(x, y)$ is just a special case of the ellipse, which we already discussed.

3. Systems of Ordinary Differential Equations

Definition 1.26. A *system of ordinary differential equations* is a set of equations involving involving a set of unknown functions $y_1,\dots,y_n$ and their derivatives each a function of one independent variable.

Remark 1.27. We will make a thorough study of systems of differential equations. For now, we provide two examples to illustrate what we mean.

Example 1.28 (Linear Cascade Equation)**.** Let $\alpha, \beta, \gamma \in \mathbb{R}$. The following system of equations is sometimes called a cascade equation:

$$\begin{aligned} y_1' &= \alpha y_1 \\ y_2' &= \beta y_1 + \gamma y_2 \end{aligned}$$

This is because the unknown functions *cascade* down the right-hand-side. These systems can be used to model certain kinds of radioactive decay, which we will discuss in detail later.

Example 1.29 (Lotka-Volterra Equations)**.** Let $x(t)$ be the quantity of a prey species (e.g., rabbits) in an environment and let $y(t)$ be the quantity of a predator species (e.g., wolves) in an environment. The Lotka-Volterra equations describe a semi-idealized model of predator-prey interaction. Let $\alpha, \beta, \gamma, \delta \geq 0$ the equations are:

$$\begin{aligned} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= \gamma xy - \delta y. \end{aligned}$$

These equations have interesting properties, which we will come to after we have dispatched with ordinary differential equations.

LESSON 2

1. Initial Value Problems

Definition 2.1 (Initial Value Problem)**.** An initial value problem (IVP) is a differential equation $F(y, y', \dots, y^{(n)}, x) = 0$ along with conditions:

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

Example 2.2 (Galilean Gravity)**.** Consider the IVP:

$$\ddot{y} = -g.$$
$$y(0) = y_0 \quad y'(0) = v_0$$

We had the solution $y(t) = C_0 + C_1 t - \frac{1}{2}gt^2$. Setting $t = 0$ we have:

$$y(0) = C_0 + C_1(0) - \frac{1}{2}g(0)^2 = y_0 \implies C_0 = y_0$$

and

$$y'(0) = C_1 - g(0) = v_0 \implies C_1 = v_0.$$

Thus the solution to the IVP is $y(t) = y_0 + v_0 t - \frac{1}{2}gt^2$.

Example 2.3 (Exponential Growth/Decay)**.** Let's solve a more complicated IVP.

$$y' - \alpha y = 0$$
$$y(0) = y_0 > 0$$

We can write the ODE as:

$$\frac{dy}{dx} = \alpha y$$

Multiplying by dx, diving by y and integrating yields:

$$\int \frac{dy}{y} = \int \alpha dx.$$

A little computation shows:

$$\log(y) = \alpha x + C,$$

were C is a constant of integration. You can put it on either side, but by convention it usually goes on the side with the independent variable. Since $y(0) = y_0$ it's easy to see that when $x = 0$ we have:

$$\log(y_0) = C$$

Thus $\log(y) - \alpha x - \log(y_0) = 0$ implicitly solves the ODE. We can do better. Notice:

$$\log(y) - \log(y_0) = \log\left(\frac{y}{y_0}\right) = \alpha x.$$

Taking the exponential yields:

$$\frac{y}{y_0} = e^{\alpha x}$$

or:

$$y(x) = y_0 e^{\alpha x}.$$

Remark 2.4. Notice if $y_0 > 0$ (as we assumed) then $y_0 e^{\alpha x} > 0$ for all x. If $y_0 < 0$ we need to use the (more correct) formula:

$$\int \frac{dx}{x} = \log|x|.$$

The result is the same. If $y_0 \in \mathbb{R}$, we will always have the solution $y(x) = y_0 e^{\alpha x}$ for the equation $y' = \alpha y$ with $y(0) = y_0$.

Example 2.5 (Radioactive Decay)**.** Some elements (especially those with high atomic number) decay into lighter elements. You're familiar from popular culture with the radioactive properties of Cesium, Plutonium and some isotopes of Uranium. Let $P(t)$ be the mass of a sample of the element at any given time. Radioactive decay is modeled by the equation:

$$\dot{P} = -kP,$$

where $k > 0$ is the decay constant. In particular, the radioactive decay law states that the rate of decrease of the elements mass at time t is proportional to its mass at time t. (More mass implies a faster instantaneous rate of decay.) We already know the resulting solution is:

$$P(t) = P_0 e^{-kt}.$$

A time of interest to chemists, environmental engineers and physicists is the *half-life*. This is the time when half of an initial sample P_0 will have decayed away. We can compute this as:

$$\frac{P_0}{2} = P_0 e^{-k\tau},$$

where τ is the half-life. Solving for τ we see:

$$\log\left(\frac{1}{2}\right) = -\log(2) = -k\tau$$

or:

$$\tau = \frac{\log(2)}{k}. \tag{2.1}$$

Notice the half-life is independent of initial sample size (as expected). Also, given a half-life we can compute a decay rate by solving for k in terms of τ.

Remark 2.6. You may remember that the equation $y(t) = y_0 e^{\alpha t}$ (here t is time) models exponential growth when $\alpha > 0$ and exponential decay when $\alpha < 0$. Exponential growth is used to model some cellular growth and the very early stages of pandemics (sigh). We have already discussed exponential decay.

2. Linear Ordinary Differential Equations

Definition 2.7 (Linear ODE)**.** An ordinary differential equation is linear if

$$F(y, y', \dots, y^{(n)}, x) = 0$$

is linear in the unknown function and its derivatives.

Remark 2.8. In particular, if $F(y, y', \dots, y^{(n)}, x) = 0$, that means y cannot be multiplied by any of its derivatives or itself or appear in complicated functions. The independent variable can appear in complicated functions that may be multiplied by y or its derivatives.

Example 2.9. The following equation is *linear*:

$$\sin(x)y''(x) + \tan(x)y'(x) + e^x y(x) = x^2. \tag{2.2}$$

This is because y and its derivatives only appear on their own, not interacting with each other except through addition. On the other hand, the equation:

$$y'' = \alpha y^2 \tag{2.3}$$

is *nonlinear*, with $\alpha \in \mathbb{R}$.

Remark 2.10. On the whole, linear equations are easier to solve than non-linear equations and have nicer properties. If $F(y, y', \dots, y^{(n)}, x) = 0$ is a linear ODE, then we can easily re-write it as:

$$y^{(n)} = p_{n-1}(x)y^{(n-1)}(x) + p_{n-2}(x)y^{(n-2)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) + q(x), \tag{2.4}$$

where $p_k(x)$ $(k = 0, \dots, n-1)$ and $q(x)$ are functions of x (and only x). We have already seen several examples of linear ODE's in this form in Examples 1.19 to 1.21.

Definition 2.11 (Homogeneous)**.** A linear ODE with form given by Eq. (2.4) is called *homogeneous* if $q(x) = 0$.

Lemma 2.12 (Linearity of the Derivative)**.** *Let $f(x)$ and $g(x)$ be two functions of a the independent variable x. If $c \in \mathbb{R}$ and n is a non-negative integer, then:*

$$\frac{d^n}{dx^n}\left[cf(x) + g(x)\right] = c\frac{d^n f}{dx^n} + \frac{d^n g}{dx^n} = cf^{(n)}(x) + g^{(n)}(x).$$

□

Remark 2.13. Lemma 2.12 says that the derivative is a *linear* operation (linear operator). We will discuss this in depth in the next section.

Theorem 2.14 (Superposition). *Suppose $y_1(x)$ and $y_2(x)$ are two solutions to a linear homogeneous ODE with form:*

$$y^{(n)} = p_{n-1}(x)y^{(n-1)}(x) + p_{n-2}(x)y^{(n-2)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x).$$

Then $\alpha y_1(x) + y_2(x)$ is also a solution.

Proof. Apply Lemma 2.12:

$$\frac{d^n}{dx^n}\left[\alpha y_1(x) + y_2(x)\right] = \alpha y_1^{(n)}(x) + y_2^{(n)}(x) =$$
$$\alpha\left[p_{n-1}(x)y_1^{(n-1)}(x) + p_{n-2}(x)y_1^{(n-2)}(x) + \cdots + p_1(x)y_1'(x) + p_0(x)y_1(x)\right] +$$
$$\left[p_{n-1}(x)y_2^{(n-1)}(x) + p_{n-2}(x)y_2^{(n-2)}(x) + \cdots + p_1(x)y_2'(x) + p_0(x)y_2(x)\right].$$

Factoring terms we have:

$$\frac{d^n}{dx^n}\left[\alpha y_1(x) + y_2(x)\right] = p_{n-1}(x)\left[\alpha y_1^{(n-1)}(x) + y_2^{(n-1)}(x)\right] +$$
$$p_{n-2}(x)\left[\alpha y_2^{(n-2)}(x) + y_2^{(n-2)}(x)\right] + \cdots + p_1(x)\left[\alpha y_1'(x) + y_2'(x)\right] +$$
$$p_0(x)\left[\alpha y_1(x) + y_2(x)\right]$$

Thus $\alpha y_1(x) + y_2(x)$ satisfies the ODE equation. □

Example 2.15. Assume $\alpha > 0$ and consider the linear ODE:

$$y'' = -\alpha^2 y$$

We can show that both $y_1(x) = \sin(\alpha x)$ and $y_2(x) = \cos(\alpha x)$ are solutions. Note:

$$\frac{d^2 y_1}{dx^2} = -\alpha\sin(\alpha x) = -\alpha^2 y_1(x)$$
$$\frac{d^2 y_2}{dx^2} = -\alpha\cos(\alpha x) = -\alpha^2 y_2(x).$$

Thus we can conclude a solution to the original ODE might have form $C_1\cos(\alpha x) + C_2\sin(\alpha x)$, where C_1 and C_2 are constants of integration.

Remark 2.16. We will see this pattern repeat for all second order linear homogeneous ODE's. There will be two solutions that can be combined via superposition with two constants of integration integration.

3. Introduction to Existence and Uniqueness

Remark 2.17. We just saw an example of a second order linear equation where seemingly two separate solutions were combined into a single solution. How do we know a solution exists at all though? This question is easiest to answer for first order equations.

Theorem 2.18 (Picard's Theorem). *Consider the first order initial value problem:*

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0.$$

If both f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R = [a, b] \times [c, d]$ containing (x_0, y_0) then there exists a $\delta > 0$ such that the IVP has a unique solution $\varphi(x)$ in some interval $(x_0 - \delta, x_0 + \delta)$. □

Remark 2.19. Interestingly, existence of a solution is guaranteed by the continuity of f alone, but not necessarily uniqueness.

Example 2.20. Consider the ODE:

$$y' = \frac{1}{2xy}.$$

Notice the function on the right-hand-side is not continuous at $x = 0$ or $y = 0$. That is, this function has infinitely large regions of discontinuity. Nevertheless, we can solve this ODE.

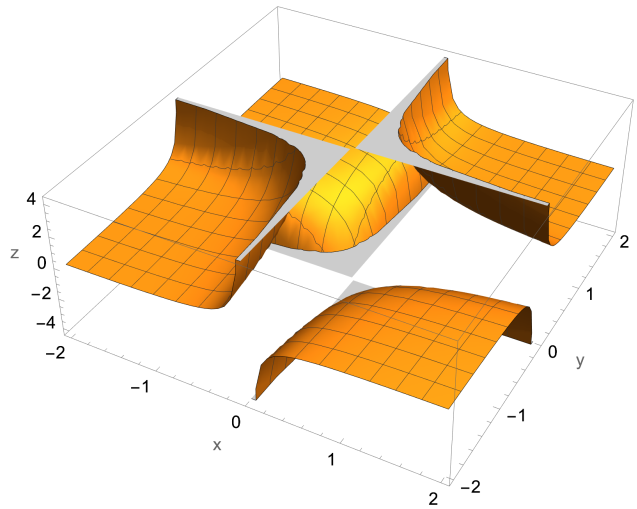


FIGURE 2.1. A function that is discontinuous along the lines $x = 0$ and $y = 0$.

Rearranging terms and integrating yields:

$$\int 2y\,dy = \int \frac{dx}{x}.$$

Therefore:

$$y^2 = \log(x) + C.$$

We know from the theorem that existence and uniqueness are only possible in a rectangle that does not contain the line $x = 0$ or the line $y = 0$. This is clear: If we start with an initial condition $y(0) = y_0$ with $y_0 \neq 0$, then this implies

$$y_0^2 = \log(0) + C,$$

but $\log(0)$ is undefined, so we can have no such solution. On the other hand, suppose we start with a solution $y(x_0) = 0$ with $x_0 > 0$. Then we have:

$$0^2 = \log(x_0) + C \implies C = -\log(x_0).$$

But $y^2 = \log(x) + C$ this implies we have two solutions:

$$y = \pm\sqrt{\log(x) - \log(x_0)},$$

which may exist in different intervals depending on value of x_0.

LESSON 3

1. Partial Differential Equations

Definition 3.1. A *partial differential equation* (PDE) is an equation involving an *unknown* function of several variables $u(x_1, \dots, x_n)$ with $n > 1$ and any number of its partial derivatives.

Remark 3.2 (Notation)**.** Notation for partial derivatives varies by text convenience. For simplicity, let $u(x, t)$ be a function of two variables. In this case, x is usually a *spatial* variable and t is usually a *temporal* (time) variable. By way of notation for derivatives we have the following equivalent expressions:

$$\frac{\partial u}{\partial t} = u_t = \partial_t u$$
$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \partial_{xx} u$$

Notation varies by author and whatever happens to be convenient. There are still other notations that can be used.

Definition 3.3 (PDE Order)**.** The *order* of a PDE is the order of the highest partial derivative appearing in the equation.

Example 3.4 (Simple Transport Equation)**.** Let $u(x, t)$ be a function of space x and time t and let $c \in \mathbb{R}$, then the equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{3.1}$$

is a partial differential equation, which we'll refer to as the *simple transport equation* because there are ways to make it more complex and thus more general.

Proposition 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^1$. Then $u(x, t) = f(x - ct)$ is a solution to the simple transport equation.*

Proof. We can differentiate and check:

$$\frac{\partial u}{\partial t} = \frac{d}{dt} f(x - ct) = -cf'(x - ct).$$

On the other hand:

$$\frac{\partial u}{\partial x} = \frac{d}{dx} f(x - ct) = f'(x - ct).$$

Therefore:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -cf'(x - ct) + cf'(x - ct) = 0$$

□

Remark 3.6. Before commenting on the previous proposition, recall that the solution to the ordinary differential equation (ODE):

$$\frac{du}{dt} = \lambda u \tag{3.2}$$

is:

$$u(t) = Ae^{\lambda t},$$

where A is a constant of integration to be determined by an initial condition on the ODE.

By contrast, when we solved the simple transport equation, we don't just have an unspecified constant we have an unspecified function, namely $f(z)$.

Example 3.7 (Physical Interpretation of the Simple Transport Equation)**.** Let us be very concrete for a moment. Suppose we have the initial condition:

$$u(x, 0) = e^{\frac{-x^2}{2}}.$$

Furthermore, suppose $c = 1$. The solution is:

$$u(x, t) = e^{\frac{-(x-t)^2}{2}}. \tag{3.3}$$

This is illustrated in Fig. 3.1. We see at once that the transport equation describes a function

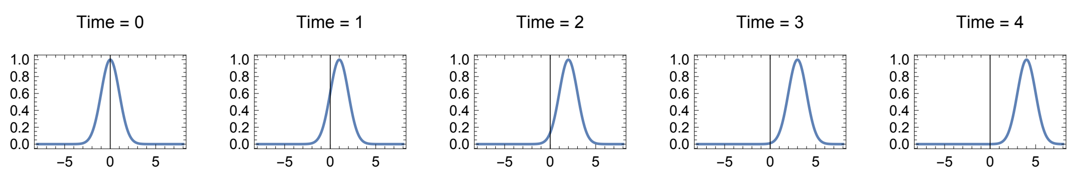


FIGURE 3.1. Solutions to the transport equation are functions traveling right.

$f(x)$ (denoting some physical system) moving to the right with speed c. In the example, the top of the curve is always at $x = t$, thus the curve is moving at speed $c = 1$ (as expected).

Example 3.8 (Heat (Diffusion) Equation)**.** Let $k > 0$. The one-dimensional *heat (or diffusion) equation* is the second order partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{3.4}$$

Proposition 3.9. *Assume $t \in (0, \infty)$ and $x \in (-\infty, \infty)$, then the function:*

$$\Phi(x, t) = \frac{e^{-\frac{x^2}{4kt}}}{2\sqrt{k\pi t}} = \frac{1}{2\sqrt{k\pi t}} \exp\left(-\frac{x^2}{4kt}\right) = \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi kt}} \tag{3.5}$$

solves the heat equation in 1 dimension.

Example 3.10 (Schrödinger Equation)**.** Related to, but distinct from, the heat equation is the one dimensional Schrödinger Equation of quantum mechanics:

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x, t)\Psi(x, t). \tag{3.6}$$

This is (essentially) a diffusion equation with an *imaginary* diffusion constant (recall $i = \sqrt{-1}$). This small change makes a big difference in the nature of the solutions. Also, unlike the heat equation, which can be derived from first principles, the Schrödinger's Equation does not seem to have a natural derivation; it simply popped out of Schrödinger's mind.

Example 3.11 (Wave Equation)**.** Let $c \in \mathbb{R}$. The one-dimensional *wave equation* is the second order partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.7}$$

Proposition 3.12. *Let $F(z)$ and $G(z)$ be twice differentiable functions with continuous derivatives. Then:*

$$u(x,t) = F(x - ct) + G(x + ct) \tag{3.8}$$

is a solution to the wave equation.

Proof. The proof is by differentiation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 F''(x - ct) + c^2 G''(x + ct)$$

and

$$\frac{\partial^2 u}{\partial x^2} = F''(x - ct) + G''(x + ct)$$

The result follows immediately. □

Remark 3.13. Note from the previous proposition we can deduce that one general solution to the wave equation is a pair of traveling waves going in opposite directions.

Definition 3.14 (Linear Partial Differential Equation)**.** A PDE is *linear* if it is a linear function of the unknown function u and all of its derivatives. Otherwise, it is called *non-linear.*

Remark 3.15. So far all the PDE's you've seen are linear. We will return to the one dimensional heat and wave equations at the end of the course. At that point, we'll impose some additional conditions that solutions must satisfy so that we do not have to deal with quite so general functions.

2. Linear Operators & Eigenfunctions

Definition 3.16 (Linear Operator)**.** An *operator* is a mapping from a function space U to a function space V. Given two elements $u, v \in U$ (resp. V), assume that $u + v \in U$ (resp. V) and for some $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$) then $\alpha u \in U$ (resp. V). An operator $L : U \to V$ is *linear* if:

$$L(\alpha u + v) = \alpha L(u) + L(v) \tag{3.9}$$

Remark 3.17. Put simply, an operator is a machine that turns one function into another function.

Example 3.18. You already know two linear operators that are critical in the study of calculus. The derivative and the integral are both linear operators. To see this, recall the following rules from calculus. Assume $f, g : \mathbb{R} \to \mathbb{R}$ are C^1 functions, then:

$$\frac{d}{dx}(f + \alpha g) = \frac{df}{dx} + \alpha\frac{dg}{dx}.$$

Likewise, if $f, g : \mathbb{R} \to \mathbb{R}$ are C^0 functions, then:

$$\int \alpha f(x) + g(x)\, dx = \alpha \int f(x)\, dx + \int g(x)\, dx$$

Example 3.19. Let's write a general linear ODE in operator notation. Let:

$$L = \left(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right)$$

Then the ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$$

can be written as:

$$L(y) = f(x),$$

because we have:

$$L(y) = \left(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right) y(x) = \frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x)$$

Remark 3.20. We have already proved (essentially) the next proposition.

Proposition 3.21 (Superposition Redux)**.** *Let L be a linear operator and suppose v and w both solve $L(u) = 0$. Then for any constants α and β we have:*

$$L(\alpha v + \beta w) = 0,$$

that is $\alpha v + \beta w$ also satisfies the equation $L(u) = 0$. □

Example 3.22. We can also generalize the idea of an operator to re-write linear PDE's in simpler form. Consider the simple transport equation and define:

$$L = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

Then for some C^1 function u, we have:

$$L(u) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}.$$

Thus, the simple transport equation can be written as:

$$Lu = 0.$$

Example 3.23 (D'Almbertian)**.** The D'Almbertian operator defines the wave equation. In one dimension we have:

$$\Box = \frac{\partial}{\partial t^2} - c^2\frac{\partial}{\partial x^2}.$$

In higher dimensions it is:

$$\Box = \frac{\partial}{\partial t^2} - c^2\Delta.$$

The wave equation is then:

$$\Box u = 0.$$

It is worth noting that in one dimension, you can factor the D'Almbertian as:

$$\Box = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right).$$

This helps to explain D'Almbert's solution to the wave equation as a pair of traveling waves, one going leftward and another going rightward.

Remark 3.24. It is worth convincing yourselves that if L is a linear operator and u and v are functions that both solve the homogeneous problem $L(u) = 0$ and $L(v) = 0$, then $u + v$ also solves the homogeneous problem $L(u + v) = 0$. Thus, the principle of superposition holds for all linear homogeneous ODE's and PDE's

Definition 3.25 (Eigenvalues & Eigenfunctions)**.** Let L be a linear operator defined on some appropriate function spaces. A function u is an eigenfunction with eigenvalue $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$) if:

$$L(u) = \lambda u \tag{3.10}$$

Remark 3.26. For those of you who have taken matrices, this is in perfect analogy to eigenvalues and eigenvectors of square matrices. Just like eigenvectors, where scale doesn't matter, we will see in the next example that constant scaling also doesn't matter for eigenfunctions.

Example 3.27. Let us return again to the exponential growth and decay ODE. Let:

$$L = \frac{d}{dx}$$

be the differential operator. Then we have:

$$y' = \lambda y \iff L(y) = \lambda y. \tag{3.11}$$

We know that the solution to this ODE is:

$$y(x) = Ae^{\lambda x}.$$

Thus $y(x) = e^{\lambda x}$ is an eigenfunction of L with eigenvalue λ. Notice, we can ignore the constant A in front, since if $y(x)$ is an eigenfunction of L, then so is $Ay(x)$ by linearity. Thus we have shown that the set of all possible real eigenvalues for L is $\mathbb{R}$. The set of all possible eigenvalues is called the *spectrum of the operator*. Notice, we're ignoring complex numbers here to make our life much easier.

Remark 3.28. While it may seem strange, eigenvalue problems pervade most of advanced ODE's and especially PDE's. While writing ODE's or PDE's in this strange form won't help you solve them, it does help in writing proofs and generalizing results. Its also helpful for understanding topics like quantum mechanics, which are almost entirely based on operator theory.

Module 2

Qualitative Analysis of First Order ODEs

LESSON 4

1. Direction Fields

Remark 4.1. Having gotten overly abstract, let's crash back to earth by drawing some pictures of ODE's. We will focus on first order ODE's for a while.

Definition 4.2 (Direction Field)**.** Suppose that $y'(x) = f(x, y)$ is a first order ODE. The direction field is a plot consisting of a line segment at each point (x, y) with small length and slope equal to $f(x, y)$.

Derivation 4.3. Consider the simple ODE $y'(x) = f(x, y)$. To draw a direction field:

(1) Choose a point (x_0, y_0)
(2) Pick a small value Δx.
(3) We don't know what Δy is, but we can compute it as:

$$\Delta y = f(x, y)\Delta x.$$

(4) Start at point $(x_0 - \Delta x/2, y_0 - \Delta y/2)$ and draw a line to point $(x_0 + \Delta x/2, y_0 + \Delta y/2)$. This ensures the slope of the line is $f(x, y)$.
(5) Goto 1.

You can repeat the process for as many lines as you like.

Example 4.4. Consider the simple ODE $y'(x) = 2x$. We know this has solution $y(x) = x^2 + C$ (by integration). We can draw the direction field and some example solution curves (for various values of C) by following the recipe above.

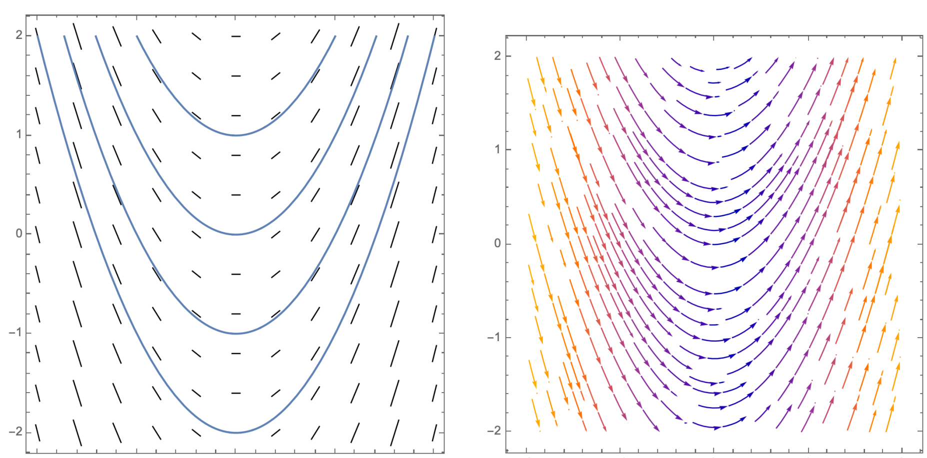


FIGURE 4.1. (Left) A direction field with example solution curves for the ODE $y'(x) = 2x$. (Right) Mathematica's stream plot of the same function.

Remark 4.5. Most modern graphing calculators can plot slope or direction fields for differential equations. In Mathematica, a direction field for the ODE $y' = f(x, y)$ can be constructed using the command `StreamPlot[f(x,y),{x,x_min,x_max},{y,y_min,y_max}]`.

Example 4.6. Consider the ODE $y'(x) = 2xy$. The direction field is shown below. From

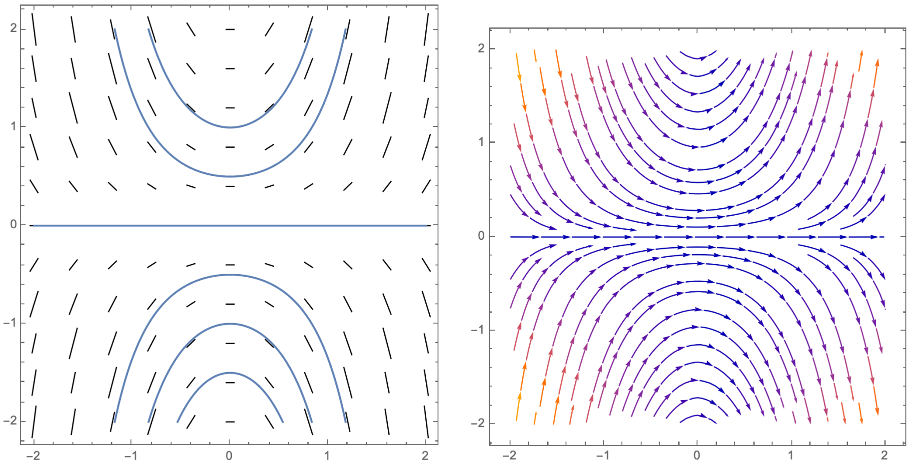


FIGURE 4.2. (Left) A direction field with example solution curves for the ODE $y'(x) = 2xy$. (Right) Mathematica's stream plot of the same function.

this figure (and the added curves) we can see that the solutions seem to come in three flavors, a constant (the line in the middle), and two types of curves that mirror each other that go off to $\pm\infty$ depending on what your initial $y(x_0) = y_0$.

This ODE is linear and can be solved in much the same way we solved the exponential growth/decay problem. We have:

$$\frac{dy}{dx} = 2xy$$

Isolating the variables on either side and integrating yields:

$$\int \frac{dy}{y} = \int 2x, dx.$$

This leads to the implicit solution:

$$\log(y) = x^2 + C.$$

Solving for y we have the general solution:

$$y(x) = A\exp(x^2),$$

where $A = \exp(C)$ is just the constant of integration in disguise. Notice that when $A = 0$ we get the constant solution $y = 0$. When $A < 0$ we get the solution curves below the line $y = 0$ and when $A > 0$ we get the solution curves above the line $y = 0$.

Example 4.7 (Discontinuous Right Hand Side)**.** Consider the ODE:

$$y' = 2y/x$$

The direction field is shown below. Notice when $x = 0$, the direction field is not shown. The field would be a collection of vertical lines. As before we can solve this ODE by isolating the two variables on opposite sides of the equals sign and integrating. We have:

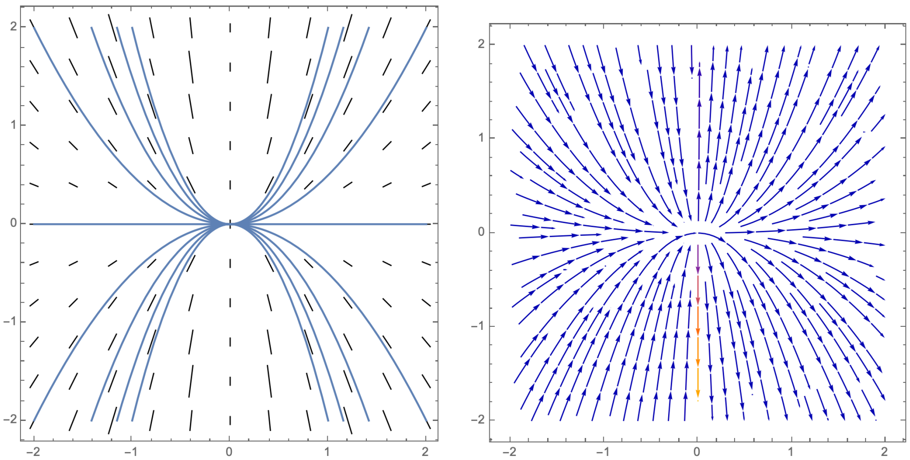


FIGURE 4.3. (Left)The direction field of the ODE $y' = 2y/x$. The right-hand-side is discontinuous at $x = 0$ leading to intersecting solutions. (Right) Mathematica's stream plot of the same function.

$$\int \frac{dy}{y} = 2\int \frac{dx}{x}.$$

This implies that:

$$\log(y) = 2\log(x) + C = \log\left(x^2\right) + C.$$

Solving for y we have:

$$y = Ax^2,$$

where $A = e^C$ is the constant of integration in a disguise. We can see the non-uniqueness of solutions if we have $y(0) = 0$. For any value of A, we will satisfy this initial condition, so there are an infinite number of solutions in this case.

Definition 4.8 (Newton's Second Law of Motion)**.** If an object of mass m is acted upon by a (vectorial) force $\mathbf{F}$, then the resulting (vectorial) acceleration $\mathbf{a}$ is given by:

$$\mathbf{a} = \frac{\mathbf{F}}{m},$$

or more familiarly: $\mathbf{F} = m\mathbf{a}$. This is *Newton's Second Law of Motion.*

Example 4.9 (Terminal Velocity)**.** Galilean gravity assumes no air resistance. In the presence of air resistance, we must model all forces on the falling body. Acceleration in this context is y''. The resulting equation of motion is given by initial value problem:

$$m\ddot{y} = mg - k\dot{y}^2 \quad y(0) = y_0, y'(0) = v_0.$$

here:

$$k = \frac{1}{2}\rho A C_d,$$

where:

A: is the cross-sectional area of the falling body,
ρ: is the density of the fluid (air) through which the body is falling and
C_d: is the drag coefficient.

Letting $\dot{y} = v$ reduces this second order ODE to a first order ODE:

$$\dot{y} = g - \frac{k}{m}v^2 \quad v(0) = v_0.$$

Without solving the equation we can analyze the equation with a direction field. We can see that depending on the starting velocity, the object will either accelerate to a terminal velocity or decelerate from a starting velocity that is too high to a terminal velocity as a result of the drag from the fluid (air). Thus we expect $\lim_{t\to\infty} v(t) = v_T$, where v_T is a terminal velocity to be determined.

To solve the ODE rewrite it as:

$$\dot{v} = g\left(1 - \alpha^2 v^2\right),$$

where:

$$\alpha^2 = \frac{k}{mg}.$$

This form will make the integral simpler. We can now isolate the variables on opposite sides of the equal sign and integrate to yield:

$$\int \frac{dv}{1 - \alpha^2 v^2} = \int g\, dt.$$

For the left-hand-side, we need to use integration by partial fractions. We have:

$$\int \frac{dv}{1 - \alpha^2 v^2} = \int \frac{dv}{(1 - \alpha v)(1 + \alpha v)} = \int \left(\frac{1}{1 + \alpha v} + \frac{1}{1 - \alpha v}\right) dv$$

Returning to the original ODE and integrating we have:

$$\frac{1}{\alpha}\log(1 + \alpha v) - \frac{1}{\alpha}\log(1 - \alpha v) = gt + C.$$

Simplifying we have:

$$\log\left(\frac{1 + \alpha v}{1 - \alpha v}\right) = \alpha g t + C.$$

This implies:

$$\frac{1 + \alpha v}{1 - \alpha v} = B\exp(\alpha g t),$$

where $B = e^C$ is still the constant of integration. We can solve for v by noting:

$$\alpha v + \alpha v B\exp(\alpha g t) = B\exp(\alpha g t) - 1.$$

Or:

$$v(t) = \frac{1}{\alpha}\frac{B\exp(\alpha g t) - 1}{B\exp(\alpha g t) + 1}.$$

At this point, we could use $v(0)$ to find B from the expression:

$$\frac{1}{\alpha}\frac{B - 1}{B + 1} = v_0,$$

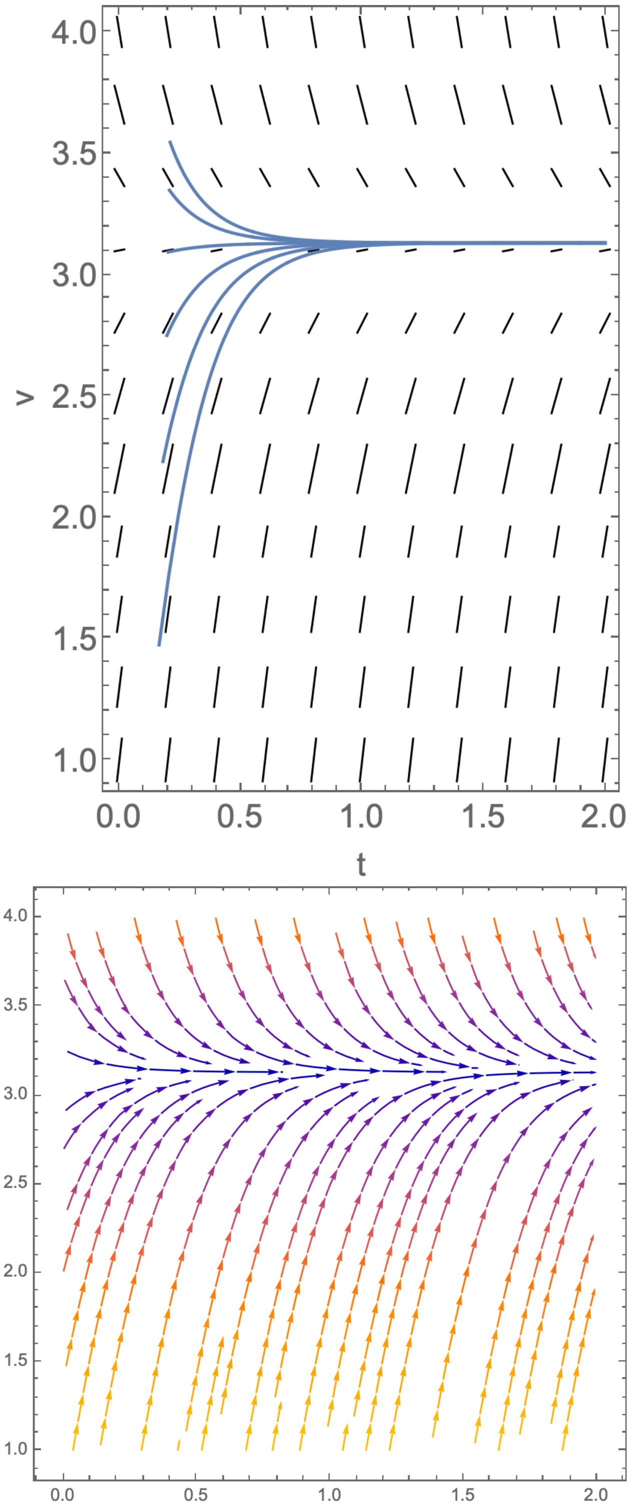


FIGURE 4.4. (Left) The direction field shows that objects moving through a fluid (like air) will either accelerate up to a terminal velocity or decelerate to a terminal velocity if their initial velocity is faster than a certain value. (Right) Mathematica's stream plot of the same function.

but we really don't care about that. We care about the terminal velocity. We know that doesn't depend on the initial velocity. Therefore, we can just compute the limit to see:

$$v_T = \lim_{t\to\infty} \frac{1}{\alpha}\frac{B\exp(\alpha g t) - 1}{B\exp(\alpha g t) + 1} = \frac{1}{\alpha} = \sqrt{\frac{mg}{k}}.$$

We can expand k to obtain:

$$v_T = \sqrt{\frac{mg}{\frac{1}{2}\rho A C_d}} = \sqrt{\frac{2mg}{\rho A C_d}}.$$

For a human moving through air, it's reasonable to set mass $m = 70kg$, density $\rho \approx 1.225kg/m^3$, area $A \approx 0.25m^2$ (assuming a height of 2 meters and a waist size of $31in \approx 0.125m$) and constant $C_d \approx 1$. Therefore, we get a (rough) estimate of the terminal velocity of a $70kg$ skydiver as:

$$\sqrt{\frac{2\cdot 70\cdot 9.8}{0.25\cdot 1.225}} \approx 67m/s \approx 150mph.$$

Experimental results range from $53m/s$ to $76m/s$. Thus our theoretical result agrees nicely with experimental results.

Remark 4.10. The terminal velocity example in the book does not apply to sky divers, who are moving too fast. We will deal with the book example in a homework. That model is called *Stokes' Drag*.

LESSON 5

1. Isoclines

Definition 5.1 (Isocline)**.** Suppose $y' = F(x, y)$. If $k \in \mathbb{R}$, an isocline is the set:

$$\Gamma_k = \{(x, y) : F(x, y) = k\}.$$

An isocline is a curve in the (x, y) plane on which the slope is constant. .

Remark 5.2. When $k = 0$, this is sometimes called a *nullcline* – but this is more common in systems of ODE's, which we will come to later.

Derivation 5.3 (Isocline Method of Drawing Direction Fields)**.** When $F(x, y)$ is simple enough to solve for the curve Γ_k, we can use the isoclines to draw a direction field.

(1) Choose a set $K = \{k_1, \dots, k_n\}$
(2) For each $k \in K$ draw the curve $F(x, y) = k$. Denote this curve Γ_k.
(3) Choose points $X_k = \{(x_1, y_1), \dots, (x_{m_k}, y_{m_k})\}$ on Γ_k.
(4) At each point $(x, y) \in X_k$ draw a small line segment with slope k.

Example 5.4. Consider the ODE:

$$y' = ax - by$$

We can see the isoclines are given by $F(x, y) = ax - by = k$ and so the curves are given by the linear equation:

$$y = \frac{a}{b}x - \frac{k}{b}.$$

Consequently the slope field is constant along lines. We illustrate this with $a = b = 1$ in the figure below. It is worth noting that this ODE cannot be easily solved by isolating the variables on opposite sides of the equal sign. The direction field (however) suggests that as $x \to \infty$ we have $y(x) \to \infty$, in a linearly.

2. Separable Equations

Definition 5.5 (Separable Equation)**.** Let $y' = f(x, y)$ be an ODE and suppose:

$$f(x, y) = p(x)q(y)$$

Then the ODE is *separable*.

Remark 5.6. We have already seen many separable equations when we studied autonomous ODE's.

THEOREM 5.7. *Suppose $y' = p(x)q(y)$ is a separable ODE. Then the general solution is given by the implicit equation:*

$$\int \frac{dy}{q(y)} = \int p(x)\,dx \tag{5.1}$$

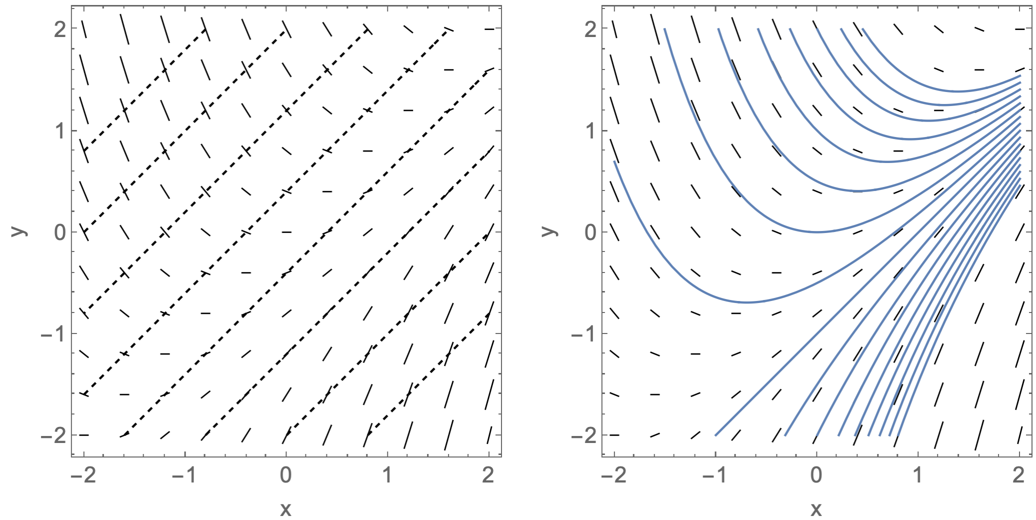


FIGURE 5.1. (Left) The isoclines of the ODE $y' = x - y$ are shown. The slope field has line segments of constant slope along lines as shown. (Right) The solution curves suggest that the function increases linear for large x.

PROOF. Define:

$$h(y) = \frac{1}{q(y)}$$

and let $H(y)$ be the antiderivative of $h(y)$ so that $H'(y) = h(y)$ and let $P(x)$ be the antiderivative of $p(x)$. Then the ODE can be written as:

$$H'(y)\frac{dy}{dx} = P'(x).$$

Suppose $y = \varphi(x)$ is a solution to the ODE. Then:

$$\frac{d}{dx}\left\{H[\varphi(x)]\right\} = H'[\varphi(x)]\frac{d\varphi}{dx} = G'(x) = \frac{d}{dx}\left[G(x)\right].$$

by the chain rule. Then it follows that as a function of x, $G(x)$ and $H[\varphi(x)]$ have the same derivative. Therefore, they must differ by at most a constant. That is:

$$H[\varphi(x)] = G(x) + C$$

for some constant C. Written implicitly this means:

$$H(y) = G(x) + C.$$

We can compute $G(x)$ explicitly since:

$$H(y) = H[\varphi(x)] = \int \frac{d}{dx}\left\{H[\varphi(x)]\right\}\,dx = \int G'(x)\,dx = \int p(x)\,dx = P(x) + C.$$

□

Remark 5.8. We are not going to do too many examples, since we've seen this solution method extensively already.

Example 5.9. Consider the initial value problem:

$$\dot{x} = \alpha e^{-t} x \quad x(0) = x_0$$

Then we have:

$$\int \frac{dx}{x} = \int \alpha e^{-t}\, dt$$

Integrating yields:

$$\log(x) = -\alpha e^{-t} + C$$

Taking the exponent yields:

$$x(t) = A \exp\left(-\alpha e^{-t}\right).$$

Using $x(0) = x_0$ we conclude:

$$x_0 = Ae^{-\alpha} \implies A = x_0 e^{\alpha}.$$

Putting it together yield:

$$x(t) = x_0 e^{\alpha} \exp\left(-\alpha e^{-t}\right) = x_0 \exp\left[\alpha(1 - e^{-t})\right].$$

Notice, unlike $\dot{x} = \alpha x$ which is unbounded if $\alpha > 0$, this function does not have that property. We have:

$$\lim_{t\to\infty} x_0 \exp\left[\alpha(1 - e^{-t})\right] = x_0 e^{\alpha}.$$

The decaying constant αe^{-t} keeps the exponential growth in check.

Example 5.10. Consider the following initial value problem:

$$y' = \frac{y-1}{x+2} \quad y(0) = 0$$

We can separate this equation to obtain:

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+2}.$$

Now we have to be careful because we will have a logarithm on both sides and we must remember to use an absolute value. Computing the integrals yields:

$$\log|y-1| = \log|x+2| + C.$$

We can solve for C now or solve for y (it won't matter). Then we have:

$$\exp\left(\log|y-1|\right) = A \exp\left(\log|x+2|\right).$$

Simplifying yields the implicit function:

$$|y-1| = A|x+2|.$$

Let's solve for A using $y(0) = 0$. We have:

$$|-1| = A|+2| \implies A = \frac{1}{2}.$$

Notice we expected A to be positive since $A = e^C$ – a fact we have more or less glossed over till now. Because $|y-1| = \pm(y-1)$ and $|x+2| = \pm(x+2)$, we must use the initial condition one more time to determine which branch of the absolute values to use. We have two options:

$$y - 1 = \frac{1}{2}(x + 2) \quad \text{or} \quad y - 1 = -\frac{1}{2}(x + 2)$$

In the first equation we see that when $x = 0$, then since $y(0) = 0$ we have $-1 = 1$, which is clearly wrong. Therefore we know we must have the solution:

$$y = 1 - \frac{1}{2}(x + 2) = -\frac{x}{2}.$$

Remark 5.11. Before moving on, it is important to note that we must keep in mind the regions where $f(x, y)$ is continuous and differentiable in y in order to know when solutions exist and are unique.

LESSON 6

1. Autonomous ODE's and Fixed Points

Definition 6.1 (Autonomous ODE)**.** A first order ordinary differential equation is *autonomous* if it can be written as:

$$y' = f(y).$$

That is, there are no independent variables explicitly on the right hand side of the equation.

Remark 6.2. We have already seen an autonomous ODE when we discussed exponential growth and decay and while discussing viscous drag on a falling body. It turns out that many systems can be modeled using autonomous ODE's. Given the ODE $y' = f(y)$, then:

$$\int \frac{dy}{f(x)} = \int dx = x + C$$

describes an implicit solution to this ODE. However, finding the integral on the left-hand-side might not be so easy. However, autonomous ODE's admit a different kind of *qualitative analysis* that is not always possible for non-autonoous ODE's.

Definition 6.3 (Fixed/Equilibrium Point)**.** A *fixed (equilibrium) point* of an autonomous ODE $y' = f(y)$ is a value y^* such that $f(y^*) = 0$ and thus if $y(0) = y^*$, then $y(x) = y^*$ for all x.

Example 6.4. Consider the drag equation:

$$\dot{v} = mg - kv^2.$$

Setting the right-hand-side equal to zero we have:

$$mg - kv^2 = 0 \implies v^* = \sqrt{\frac{mg}{k}} = \sqrt{\frac{2mg}{\rho A C_d}},$$

since:

$$k = \frac{1}{2}\rho A C_d.$$

Thus, a skydiver who begins falling at terminal velocity will experience no acceleration since $\dot{v} = 0$.

Example 6.5 (Logistic Growth)**.** Exponential growth of a species is non-physical in the long-run (even for viruses or bacteria). A more accurate model of growth is the logistic equation. Let $x(t)$ be the quantity of a population at time t. The population experiences logistic growth if we have:

$$\dot{x} = \alpha x(N - x),$$

where N is the *carrying capacity* of the population and α is the growth rate. The carrying capacity is the maximum number of individuals an environment can support. This ODE is autonomous and so we can find the fixed points:

$$x(N-x) = 0 \implies x^* = 0,\ x^* = N.$$

This ODE has two fixed points. By drawing a direction field we can see that logistic growth causes all initial populations to approach the carrying capacity. We can prove this analyti-

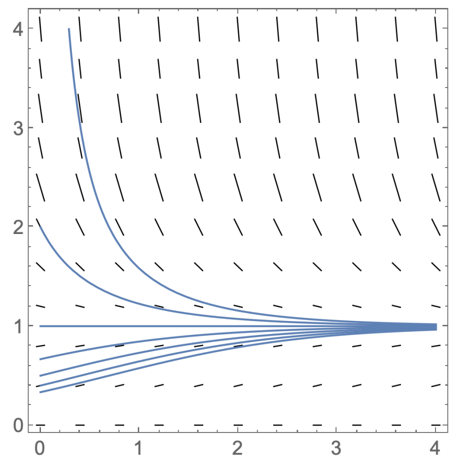


FIGURE 6.1. Logistic growth causes all initial population sizes (except the zero population) to approach the carrying capacity. In this figure, $N = 1$.

cally by solving the ODE. We have:

$$\int \frac{dy}{y(N-y)} = \int \alpha dt$$

Applying integration by partial fractions we have:

$$\int \frac{dy}{y(N-y)} = \int \frac{1}{N}\left(\frac{1}{y} + \frac{1}{N-y}\right) dy = \frac{1}{N}\left[\log(y) - \log(N-y)\right].$$

Simplifying yields:

$$\log\left(\frac{y}{N-y}\right) = \alpha N t + C.$$

Notice we are safe in not using absolute values since we assume $y > 0$ and we're tacitly assuming $y < N$. Solving for y yields:

$$\frac{N}{y} = Ae^{-\alpha N t} + 1 \implies y(t) = \frac{N}{1 + Ae^{-\alpha N t}},$$

where A is the constant of integration in disguise. If $y(0) = y_0$ we can see:

$$y_0 = \frac{N}{1+A} \implies \frac{N - y_0}{y_0} = A.$$

Taking the limit:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{N}{1 + Ae^{-\alpha N t}} = \frac{N}{1} = 1,$$

no matter what the value of A. Thus we have confirmed out assertion that all population sizes approach the carrying capacity.

2. Stability of Fixed Points

Remark 6.6. Given an IVP $\dot{y} = f(y)$ with $y(0) = y_0$, let $\varphi(t; y_0)$ denote the solution to the IVP. That is, $\varphi(0; y_0) = y_0$ and $\dot{\varphi} = f(\varphi)$.

Definition 6.7 (Stable Fixed Point)**.** Consider an autonomous ODE $\dot{y} = f(y)$ with initial value $y(0) = y_0$. A fixed point y^* is *stable* if for all $\epsilon > 0$ there is a $\delta > 0$ so that if $|y_0 - y^*| < \delta$, then $|\varphi(t, y_0) - y^*| < \epsilon$ for all $t \geq 0$.

Remark 6.8. The previous definition is not very intuitive. What it means is that a fixed point is stable if solutions tend to stay near the fixed point if they start near the fixed point, where "near" is a function of the system in question.

Definition 6.9 (Asymptotically Stable Fixed Point)**.** Consider an autonomous ODE $\dot{y} = f(y)$ with initial value $y(0) = y_0$. A fixed point y^* is *asymptotically stable* if it is stable *and* there is some $\hat{\delta} > 0$ so that if $|y_0 - y^*| < \hat{\delta}$ we have $\lim_{t\to\infty} \varphi(t; y_0) = y^*$.

Remark 6.10. This means if you start close enough to the fixed point, you not only stay near it you approach it as time goes to infinity. A fixed point is *globally asymptotically stable* if no matter what y_0 is, all solutions tend to y^* as $t \to \infty$.

Example 6.11. We have already seen two asymptotically stable fixed points. The point $y^* = N$ is an asymptotically stable fixed point in the logistic growth equation and $y^* = \sqrt{mg/k}$ is the globally asymptotically stable terminal velocity in the drag equation.

Remark 6.12. We have also encountered another kind of fixed point, namely $y^* = 0$ in the logistic growth equation. This fixed point is certainly not stable. In fact, from Fig. 6.1 all solutions that start near $y^* = 0$ seem to "run away" from that point.

Definition 6.13 (Unstable Fixed Points)**.** Consider an autonomous ODE $\dot{y} = f(y)$ with initial value $y(0) = y_0$. A fixed point y^* is *unstable* if it is not stable. It is asymptotically unstable if there is some $\hat{\delta}$ so that $|y_0 - y^*| < \hat{\delta}$ we have $\lim_{t\to-\infty} \varphi(t; y_0) = y^*$.

Example 6.14. The point $y^* = 0$ is asymptotically unstable. We can prove this. We have already seen that this solution is unstable (by evaluating the slope field in Fig. 6.1). Compute:

$$\lim_{t\to-\infty} y(t) = \lim_{t\to-\infty} \frac{N}{1 + Ae^{-\alpha N t}} = 0.$$

Remark 6.15. Deducing that $y^* = N$ was stable and $y^* = 0$ was unstable in the logistic growth equation required solving the ODE and then taking a limit. Our goal is to find a simpler way to do this.

LESSON 7

1. More on Stability

Lemma 7.1. *If $\dot{y} = \alpha\,(y - a)$, then $y^* = a$ is stable if $\alpha < 0$ and $y^* = a$ is unstable if $\alpha > 0$.*

PROOF. Let $u = y - a$. Then $\dot{u} = \dot{y}$ and we can replace the ODE with an equivalent one: $\dot{u} = \alpha u$. We know the solution here is $u(t) = Ae^{\alpha t}$. We can now replace $u(x)$ with $y(x)$ to see:

$$y(t) = a + Ae^{\alpha t}$$

If $y(0) = a$, then $A = 0$ and $y(t) = a$ for all t, so a is a fixed point as required. Now if $\alpha < 0$, then:

$$\lim_{t\to\infty} a + Ae^{\alpha t} = a,$$

now matter what A is. Likewise, if $\alpha > 0$ then:

$$\lim_{t\to-\infty} a + Ae^{\alpha t} = a.$$

This completes the proof. □

Remark 7.2. Note, this lemma says *nothing* if α is zero and that is by design.

Derivation 7.3. Consider the autonomous ODE $\dot{y} = f(y)$ and suppose y^* is a fixed point of $f(y)$; that is, $f(y^*) = 0$. Let us compute the Taylor expansion of $f(y)$ about y^*. We have:

$$f(y) = f(y^*) + f'(y^*)(y - y^*) + o(y^2) = f'(y^*)(y - y^*) + o(y^2).$$

Here $o(y^2)$ just means there are higher order terms containing y^n with $n \geq 2$. We now assume (with no justification at all) that *close* to the fixed point, we have:

$$\dot{y} \approx f'(y^*)(y - y^*).$$

Letting $\alpha = f'(y^*)$, we can see from Lemma 7.1 that if $f'(y^*) > 0$, then $y = y^*$ is unstable and if $f'(y^*) < 0$ then $y = y^*$ is stable.

We can justify this geometrically by looking at the logistic growth equation $\dot{y} = y(1-y)$. We know that $y^* = 0$ and $y^* = 1$. We can see that for $y > 1$, the lines in the slope field all have negative slope. Thus the function is being driven back down toward $y = 1$. For $y \in (0, 1)$ the lines in the slope field all have positive slope. Thus the function is being driven up from $y = 0$ to $y = 1$. The arrows on the left of Fig. 7.1(Left) illustrate this. The slopes at the fixed point along with the function $f(y) = y(1 - y)$ and the direction the function is being driven is shown by the arrows. These arrows are identical to the ones appearing on the ordinate in Fig. 7.1(Left).

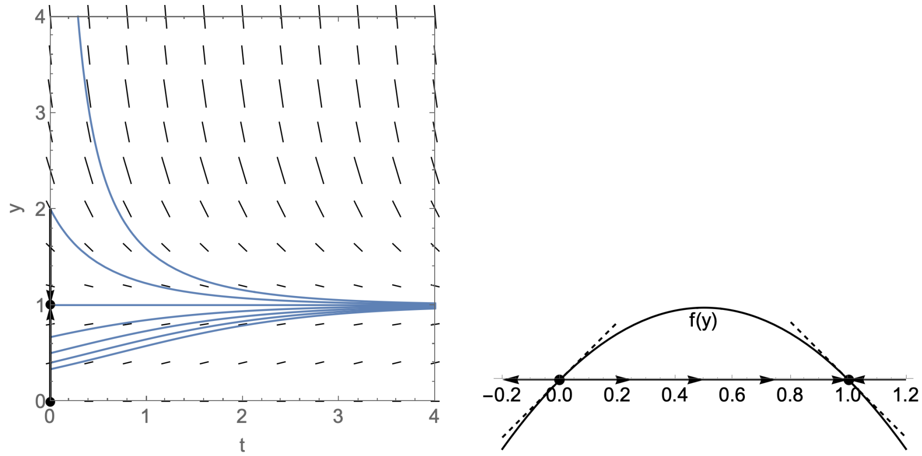


FIGURE 7.1. (Left) The slope field showing the Logistic Growth equation with arrows showing the direction $y(t)$ is being driven on the ordinate. (Right) The function $f(y)$ is shown along with tangent lines (and hence slopes) at the fixed points.

THEOREM 7.4. *Consider the autonomous ODE $y' = f(y)$ with fixed point y^*. If $f'(y^*) > 0$, then $y = y^*$ is unstable. If $f'(y^*) < 0$, then $y = y^*$ is stable. If $f'(y^*) = 0$, then no information can be obtained from f'.* □

Example 7.5. Consider the ODE $\dot{x} = x^2$. This ODE has a single fixed point $x^* = 0$, but notice $f(x) = x^2$ and $f'(x) = 2x$, so $f'(0) = 0$. So we can deduce nothing about the stability of this fixed point. This is sensible. The function $f(x) = x^2$ does not have a linear approximation by Taylor series at $x = 0$.

We can solve this ODE to see:

$$\int \frac{dx}{x^2} = \int dt \implies -\frac{1}{x} = t + C.$$

If $x(0) = x_0$ (and $x_0 \neq 0$) we have $C = -1/x_0$. Thus:

$$x(t) = \frac{1}{\frac{1}{x_0} - t} = \frac{x_0}{1 - x_0 t}.$$

The last expression clears up the issue with $x_0 = 0$ nicely. Now let us suppose $x_0 > 0$. Notice that when $t = 0$ we have $x(0) = x_0$. But as t approaches $t^* = 1/x_0$, we see the denominator approaches 0 (from the right). Thus:

$$\lim_{t \to t^{*-}} \varphi(t; x_0) = \infty.$$

Thus, the solution does not stay close to $x^* = 0$ when $x_0 > 0$, so $x^* = 0$ is not stable. In fact, it exhibits a property called *blow up* because it goes off to infinity in finite time.

However, when $x_0 < 0$ then $1 - x_0 t > 0$ for all t, so:

$$\lim_{t \to t^*} \varphi(t; x_0) = 0.$$

Consequently, $x^* = 0$ appear stable from the left. This is an equilibrium of mixed type. In a sense, it is stable in one direction and unstable in another direction. However, it is still considered unstable.

2. Potential Functions

Definition 7.6 (Potential Function). If $y' = f(y)$, then the potential function $V(y)$ has the property that:

$$f(y) = -\frac{dV}{dy}.$$

Int computing $V(y)$ as the negative of the antiderivative of $f(y)$, we always set the arbitrary constant to 0.

Example 7.7. Consider the simple ODE:

$$y' = 3y^2 - 3.$$

Then the potential function $V(y) = 3y - y^3$. Notice the minima and maxima of $V(y)$ correspond to the fixed points of the ODE.

THEOREM 7.8. *Suppose $y' = f(y)$ is an autonomous ODE with potential function $V(y)$. Let y^* be a fixed point of the ODE.*

(1) *If y^* is a local minimum of $V(y)$, then y^* is stable.*
(2) *If y^* is a local maximum of $V(y)$, then y^* is unstable.*

PROOF. We will prove the first assertion. The proof of the second assertion is nearly identical. Since $V(y)$ is an antiderivative of $f(y)$ it is straight forward to see that if y^* is a local minimum of $V(y)$, then $-V'(y^*) = f(y^*) = 0$. Moreover, since y^* is a local minimum, it follows that $V''(y) > 0$ by the second derivative test, which implies that $-V''(y^*) = f'(y^*) < 0$. The fact that y^* is stable follows from Theorem 7.4. □

Remark 7.9. Potential functions are ubiquitous in physics and engineering, where they are usually more general. In fact, gravity and electromagnetism can both be phrased and studied in terms of their (common) potential functions. While this result doesn't add much to our study of fixed points, they can be useful to the study of fixed points even when the nature of the fixed point cannot be analyzed with derivative information.

3. Bifurcations - Part 1

Derivation 7.10 (Saddle-Node Bifurcation). Consider the ODE:

$$y' = r + y^2.$$

We have three cases:

(1) If $r < 0$, then the ODE has two fixed points: $y^* = \pm\sqrt{-r}$. We know $f'(y) = 2y$.
   (a) When $y^* = \sqrt{-r}$, then $f'(y^*) = 2\sqrt{-r} > 0$ and $y^* = \sqrt{-r}$ is *unstable*.
   (b) When $y^* = -\sqrt{-r}$, then $f'(y^*) = -2\sqrt{-r} < 0$ and $y^* = -\sqrt{-r}$ is *stable*.
(2) If $r = 0$, then the ODE has one fixed point. We have already seen that $y^* = 0$ has mixed stability.
(3) When $r > 0$, then $r + y^2 = 0$ has no real solutions and there are no (real) fixed points.

We can summarize these results in a table: Thus we see that the sign of the parameter r has a serious impact on the qualitative nature of the solution curves (slope fields) we expect to see. This is called a *bifurcation* and the point of bifurcation occurs at $r = 0$ because the system goes from having two fixed points when $r < 0$ to having no fixed points when $r > 0$.

Sign of r	**Fixed Points**
$r < 0$	$y^* = \sqrt{-r}$ (unstable), $y^* = -\sqrt{-r}$ (stable)
$r = 0$	$y^* = 0$ (mixed stability)
$r > 0$	No fixed points

TABLE 1. A table summarizing the impact the value of r has on the fixed points and their stability of the ODE $y' = \dot{r} + y^2$.

Definition 7.11 (Bifurcation)**.** Let $y' = f(y; r)$, where r is a parameter not a variable. A bifurcation occurs at $r = r^*$ if the nature or number of fixed points of the autonomous ODE $y' = f(y; r^-)$ is different from the nature or number of fixed points of the autonomous ODE $y' = f(y; r^+)$ with $r^- < r^* < r^+$.

Definition 7.12 (Saddle-Node Bifurcation)**.** An ODE $y' = f(y; r)$ exhibits a saddle-node bifurcation if it can be transformed into an equivalent ODE $u' = r + u^2$ or near the critical value r^* and fixed point x^* well approximated (in the sense of Taylor) by $u' \approx r^* + u^2$. This is called the *normal form* of the bifurcation.

Example 7.13 (Quota Harvesting)**.** The following example comes from [**Arn92**]. Consider a population that grows according to a logistic equation but is periodically harvested at a constant rate c (called the quota). Population dynamics are given by the ODE:

$$\dot{x} = \alpha x(N - x) - c = -\alpha x^2 - \alpha N x - c$$

The fixed points of the system are:

$$x^* = -\frac{\alpha N \pm \sqrt{\alpha^2 N^2 - 4\alpha c}}{2\alpha} \tag{7.1}$$

When:

$$c < \frac{\alpha N^2}{4},$$

there are two fixed points, one stable and the other unstable. On the other hand, if

$$c > \frac{\alpha N^2}{4}$$

there are no real fixed points. This is a saddle-node bifurcation. We can put this into normal form by first completing the square:

$$\alpha x(N - x) - c = -\alpha \left(x + \frac{N}{2}\right)^2 - \left(c - \frac{\alpha N^2}{4}\right)$$

Let:

$$s = c - \frac{\alpha N^2}{4}$$
$$v = x + \frac{N}{2}$$

Then $\dot{v} = \dot{x}$ and we can write:

$$\dot{v} = -\alpha v^2 - s$$

Now let $u = \alpha v$ so that $\dot{u} = -\alpha\dot{v}$. Thus:

$$-\frac{1}{\alpha}\dot{u} = -\alpha\frac{1}{\alpha^2}u^2 - s = -\frac{1}{\alpha}u - s.$$

Simplifying:

$$\dot{u} = u^2 + s\alpha.$$

Let $r = s\alpha$ to see we have the equivalent system:

$$\dot{u} = u^2 + r.$$

Thus, this is a Saddle-Node bifurcation.

Example 7.14 (Quota Harvesting Continued)**.** For simplicity let $\alpha = N = 1$. Using Eq. (7.1), we see that if $c > \frac{1}{4}$, there are no fixed points. Moreover, since $x(1-x) - c < 0$ for all x when $c > \frac{1}{4}$, we know that $\dot{x} < 0$ and thus the population will crash as a result over over-harvesting. On the other hand, if $c < \frac{1}{4}$, then there is a single stable equilibrium point and the population will approach that level with responsible harvesting.

Derivation 7.15 (Bifurcation Diagram)**.** It is convenient to convey information about a bifurcation graphically since Table 1 is nice, but a bit dense. To that end, note that in $y' = r + y^2$, the fixed point(s) y^* are functions of r. Therefore, we could draw a graph of $y^*(r)$. In the case of the Saddle-Node bifurcation, we know that:

$$y^*(r) = \pm\sqrt{-r}.$$

This is a relation, not a function. However, we can plot the two fixed point values (in terms of r), using a dashed line for the unstable fixed point and a solid line for the stable fixed point. The point at $r = 0$ is shown with a half-shaded circle to indicate the fact that the

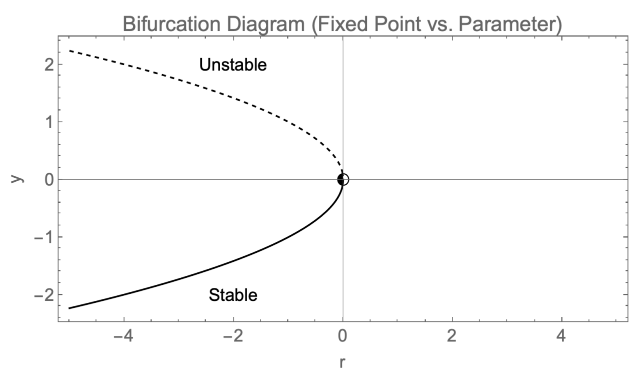


Figure 7.2. A bifurcation diagram for a saddle-node bifurcation. The parameter r varies on the abcissa, while the resulting fixed point (y^*) varies on the ordinate. The two curves show the fixed points for varying values of r and the line type shows their stability. The point mixed equilibrium at $r = 0$ is shown with a half-shaded circle.

equilibrium at $r = 0$ is mixed (stable on the left and unstable on the right). This picture is called a *bifurcation diagram*.

LESSON 8

1. Bifurcations - Part 2

Example 8.1 (Another Saddle-Node Bifurcation*)**.** A version of this example comes from [DD92]. Consider the ODE:

$$\dot{x} = ae^x - x.$$

The resulting fixed point equation $ae^x - x = 0$ cannot be easily solved, but we can plot the two functions ae^x and x and see where they intersect. Notice as a approach $a^* = 1/e$ the

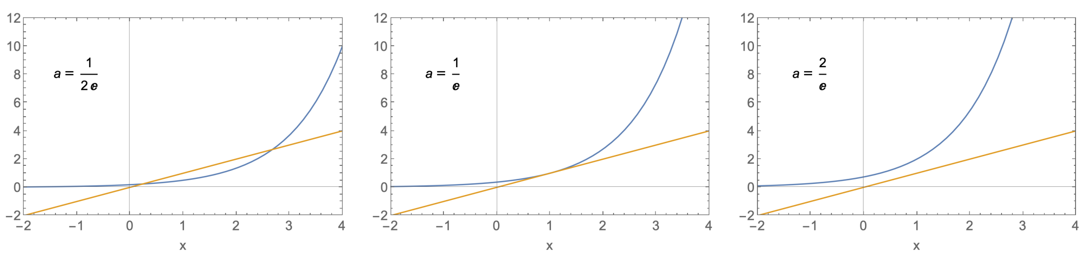


FIGURE 8.1. An unusual example of a Saddle-Node bifurcation results from the curve $f(x) = ae^x$ moving to intersect the lines $g(x) = x$ for different values of a.

line $g(x) = x$ becomes a tangent line with a single point of intersection $x = 1$. If $a < a^*$, there are two fixed points corresponding to the two points of intersection between $f(x)$ and $g(x)$. If $a > a^*$ there are no equilibrium points because $f(x)$ and $g(x)$ do not intersect. A little analysis shows that when there are two fixed points, one is stable and the other is unstable. Moreover, using a numerical solver we can construct a bifurcation diagram. The constructed bifurcation diagram looks like a distorted form of Fig. 7.2, which is enough enough to argue that this is just a saddle-node bifurcation. However, performing a Taylor expansion (in two-dimensions) around the point $a^* = 1/e$, $x^* = 1$ leads to the second order approximation:

$$ae^x - x \approx \frac{1}{2}ea\left(x^2 + 1\right) - x$$

Using appropriate scaling (as in Example 7.13), we can re-write the right-hand-side as $\dot{u} = u^2 + r$ for a constant r and thus, this is a saddle-node bifurcation.

Remark 8.2. For the remaining bifurcations, we will not go into quite so much detail (especially in terms of computing normal forms).

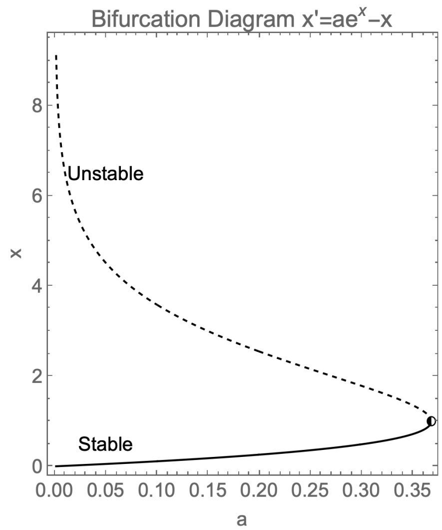


FIGURE 8.2. A numerically constructed bifurcation diagram for $\dot{x} = ae^x - x$ shows a saddle-node bifurcation.

Derivation 8.3 (Transcritical Bifurcation)**.** Consider the ODE:

$$y' = ry - y^2 = y(r - y). \tag{8.1}$$

This is very similar to the logistic growth equation except now we allow r to take on any real value as opposed to positive values (when it is a carrying capacity). The fixed points of the ODE are $y^* = 0$ and $y^* = r$. Notice when $r = 0$ there is only one fixed point namely $y^* = 0$. We can compute:

$$\frac{d}{dy}\left[y(r - y)\right] = r - 2y.$$

Using this information we have the following bifurcation table: The case when $y^* = r = 0$

Sign of r	**Fixed Points**
$r < 0$	$y^* = 0$ (stable), $y^* = r$ (unstable)
$r = 0$	$y^* = 0$ (mixed stability)
$r > 0$	$y^* = 0$ (unstable), $y^* = r$ (stable)

TABLE 1. A table summarizing the impact the value of r has on the fixed points and their stability of the ODE $y' = y(r - y)$.

follows from the fact that $y' = -y^2$, which is just a variation on the ODE $y' = y^2$, which we have already studied. From the table, we see that the fixed points *switch stability* at $r^* = 0$, though the fixed point $y^* = 0$ is always present. The bifurcation diagram shows this switch. This bifurcation is called a *transcritical bifurcation* with normal form given in Eq. (8.1).

Example 8.4 (Simple Laser*)**.** This example is discussed in [Str18]. A laser consists of a cavity filled with a *gain medium* (Helium-Neon gas, for example), a mirror for reflection and an *optical coupler* that allows light of a certain frequency to pass through. A simplified model of the behavior of the laser can be thought of as follows: As energy is applied to the gain medium (called pumping), electrons in orbit of the gain medium atoms become excited

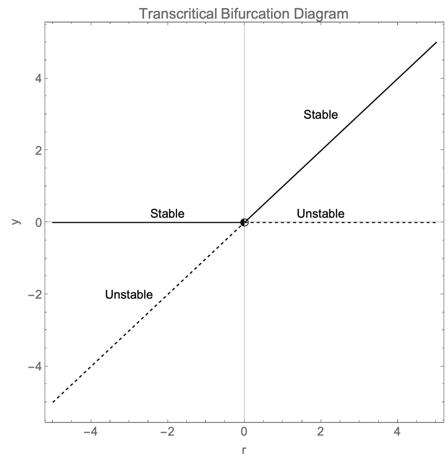


FIGURE 8.3. A bifurcation diagram for a transcritical bifurcation (in normal form). Notice the the two fixed points $y^* = 0$ and $y^* = r$ swap stability at $r^* = 0$.

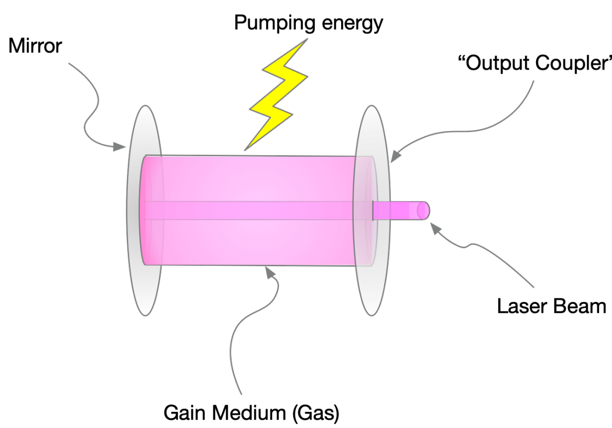


FIGURE 8.4. A laser consists of a substance that emits light when stimulated by energy (an electric field). At a certain level of stimulation, the light emitted reaches a coherent frequency and is emitted as a focused beam, unlike a light bulb which is emitted in all directions and frequencies.

and increase their energy level. As they fall back to a lower energy orbit, they emit photons. Let $n(t)$ be the number of photons in the laser cavity and let $N(t)$ the number of excited atoms in the cavity. When a photon and an excited atom interact, this causes the emission of a new photon. At any time, a proportion of these photons exits (through the optical coupler). We can write this as:

$$\dot{n} = GNn - kn \tag{8.2}$$

Here $G > 0$ is called the gain coefficient (depending on the gain medium), $k > 0$ is the constant of proportionality that governs the loss of photons and $N(t)n(t)$ models the interaction of excited atoms and photons.

If we assume that the pumping energy keeps the total number of excited atoms constant (in the absence of any interactions with photons), then we have

$$N(t) = N_0 - \alpha n(t). \tag{8.3}$$

Here N_0 is the number of excited atoms caused by the pumping. These excited atoms are lost because of the emission of photons, which we model as $\alpha n(t)$, where α is another rate constant. Combining Eqs. (8.2) and (8.3) we have:

$$\dot{n} = G(N_0 - \alpha n)n - kn = (GN_0 - k - G\alpha n)n$$

Let's simplify this equation by writing:

$$\beta = G\alpha$$
$$r = \frac{GN_0 - k}{G\alpha}.$$

Then the equation becomes:

$$\dot{n} = \beta n(r - n).$$

There are two fixed points $n^* = 0$ and $n^* = r$. As in the normal form of the transcritical bifurcation when $r < 0$, $n^* = 0$ is stable. That is, the number of photons in the laser cavity will approach zero and the apparatus will act like a lamp. However, when $r > 0$ then $n^* = r$ is stable and the apparatus maintains a constant number of photons in the laser cavity. This coherence is what produces a laser. In particular we have:

$$\frac{GN_0 - k}{G\alpha} > 0 \implies N_0 > \frac{k}{G}.$$

Thus, as we increase the pumping energy N_0, we reach a threshold (bifurcation) where the apparatus spontaneously becomes a laser (instead of a lamp). The value $N_0 = k/G$ is called the *laser threshold*.

2. Bifurcations - Part 3

Derivation 8.5 (Supercritical Pitchfork Bifurcation)**.** Consider the ODE:

$$y' = ry - y^3 = y(r - y^2).$$

This ODE has three fixed points when $r > 0$: $y^* = 0$ and $y^* = \pm\sqrt{r}$. Since:

$$\frac{d}{dy}\left[y(r - y^2)\right] = r - 3y^2, \tag{8.4}$$

we can see in this case that $y^* = 0$ is unstable while $y^* = \pm\sqrt{r}$ must be stable.

When $r = 0$ there is only one fixed point $y^* = 0$ and the ODE becomes:

$$y' = -y^3.$$

Solving this ODE (as we have done in the past) yields:

$$\int \frac{dy}{y^3} = \int -dx,$$

or

$$-\frac{1}{2y^2} = -x + C.$$

If $y(0) = y_0$, then $C = -(2y_0^2)^{-1}$. We can solve for $y(x)$ as:

$$y^2 = \frac{1}{\frac{1}{y_0^2} + 2x}.$$

If $y_0 > 0$, we use the positive branch to see:

$$y = \frac{1}{\sqrt{\frac{1}{y_0^2} + 2x}}.$$

As $x \to \infty$, $y \to 0$ and so $y^* = 0$ must be stable. If $y_0 < 0$ then we use the negative branch to see:

$$y = -\frac{1}{\sqrt{\frac{1}{y_0^2} + 2x}}.$$

However as $x \to \infty$, $y \to 0$ and so $y^* = 0$ must be stable.

Finally when $r < 0$, there is only one fixed point $y^* = 0$ and it must be stable by Eq. (8.4). Two of these three cases are illustrated using slope fields. We can use this information to

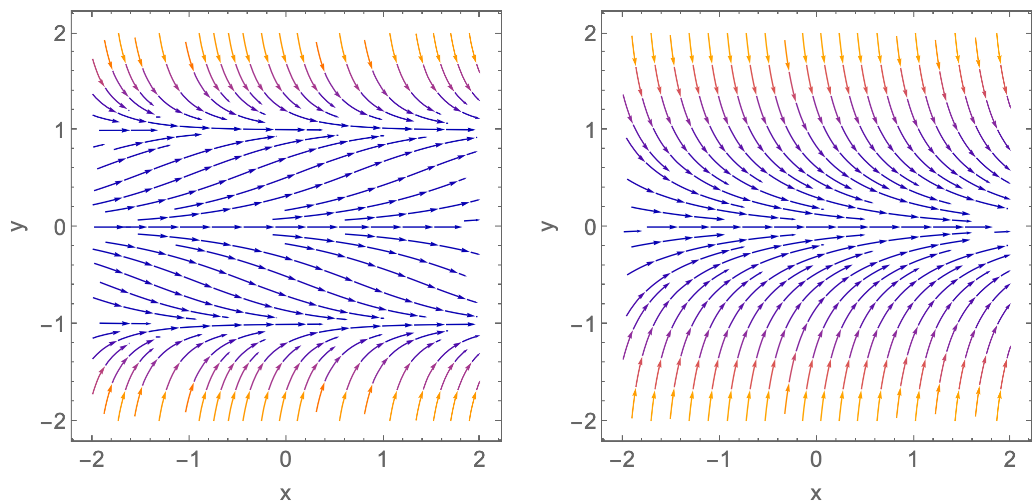


FIGURE 8.5. (Left) Direction field for the case when $r > 0$ showing two stable fixed points at $y^* = \pm\sqrt{r}$. Her $r = 1$. (Right) Direction field for the case when $r < 0$ showing one stable fixed point $y^* = 0$.

construct a bifurcation chart, that looks like a pitchfork.

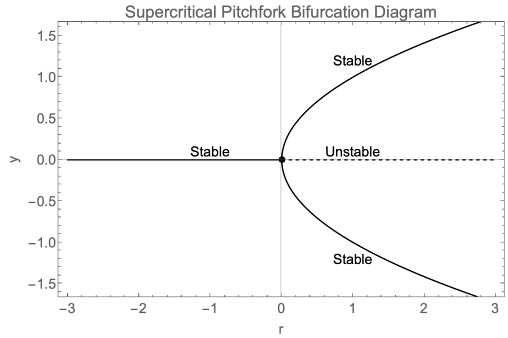


FIGURE 8.6. The bifurcation diagram, which looks like a pitchfork.

LESSON 9

1. Bifurcations - Part 4

Derivation 9.1 (Subcrcritical Pitchfork Bifurcation)**.** Consider the ODE:

$$y' = y(r + y^2).$$

Not surprisingly, this also exhibits a pitchfork bifurcation, but its behavior is different. As an exercise, you can verify the following table describing the fixed points. This leads to a

Sign of r	**Fixed Points**
$r < 0$	$y^* = 0$ (stable), $y^* = \pm\sqrt{-r}$ (unstable)
$r = 0$	$y^* = 0$ (unstable-blowup)
$r > 0$	$y^* = 0$ (unstable)

TABLE 1. A table summarizing the impact the value of r has on the fixed points and their stability of the ODE $y' = y(r + y^2)$.

bifurcation diagram that looks like a different kind of pitchfork, which you should also verify.

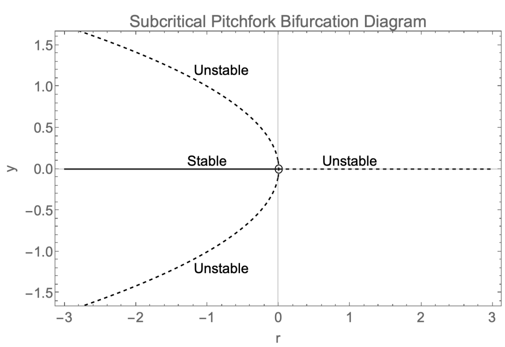


FIGURE 9.1. The bifurcation diagram, which looks like a different kind of pitchfork.

Example 9.2 (Magnetizing a Metal*)**.** The mathematics of this example is covered in [**Str18**], but the physics is nicely discussed in [**Aro11**]. The Ising-Curie-Weiss model of magnetism is crude but illustrative of the action of putting an magnetizable object into a magnetic field. (Have you ever been to an inexpensive restaurant and noticed the silverware is a little magnetic? It's been magnetized by strong magnets kept in the trash cans to keep silverware from being thrown out.) Let M be the magnetization (a measurement of how magnetic something is) of an object. Let H be an externally applied magnetic field. Usually these are vector quantities, but we're working in one dimemsion.

The *free energy* (essentially a measure of how much reversible work a system can do) of the system (sample and magnetic field) can be neatly approximated as a function of the magnetic field and the magnetization as:

$$f(M,H) \approx -HM + \frac{1}{2}aM^2 + \frac{1}{4}bM^4 + \cdots$$

Here a and b are constants that depend on the temperature. This is sometimes called the Landau expansion.

As you may have noticed, nature likes things to be in minimal energy states, so the dynamics of the *magnetization* can be modeled as:

$$\dot{M} = -\Gamma \frac{\partial f}{\partial M},$$

where $\Gamma > 0$. This is a mathematical way of saying that the magnetization of the sample will keep changing until the free energy reaches a *local minimum* in terms of M – because then $\partial_M f = 0$. Using this, we can show that free energy is a decreasing function of time (as we might intuit), but we don't need this fact.

More simply we have:

$$\dot{M} = H\Gamma - a\Gamma M - b\Gamma M^3.$$

We can get rid of some of these coefficients by letting:

$$\begin{aligned} u &= \sqrt{b\Gamma}M \\ r &= -a\Gamma \\ h &= \sqrt{b\Gamma^3}H. \end{aligned}$$

Then:

$$\dot{u} = \sqrt{b\Gamma}\dot{M} = \sqrt{b\Gamma}\left(H\Gamma + r\frac{1}{\sqrt{b\Gamma}}u - b\Gamma\frac{1}{\sqrt{b\Gamma}^3}u^3\right).$$

Notice:

$$\begin{aligned} &\sqrt{b\Gamma}(H\Gamma) = \sqrt{b\Gamma}(\sqrt{\Gamma^2}H) = \sqrt{b\Gamma^3}H = h \\ &\sqrt{b\Gamma}\left(r\frac{1}{\sqrt{b\Gamma}}\right) = r \\ &\sqrt{b\Gamma}\left(b\Gamma\frac{1}{\sqrt{b\Gamma}^3}\right) = 1 \end{aligned}$$

This leads to the ODE:

$$\dot{u} = h + ru - u^3,$$

with potential function $V(u) = \frac{1}{4}u^4 - \frac{1}{2}ru^2 - hu$. This is just a scaled version of free energy (which we are trying to minimize). Thus, free energy is (up to scale) just a potential function.

When $h = 0$, clearly this exhibits a supercritical pitchfork bifurcation. However, that assumes the external magnetic field is zero (or $b = 0$, which is unlikely). We can study this *symmetry broken* or *imperfect* bifurcation. Finding fixed points of the function $f(u) = h + ru - u^3$ is non-trivial as we know there may be 1, 2 or 3 roots. However, we can study the two functions $h + ru$ and u^3. From Fig. 9.2 we can see that for fixed h as r increases there is a special r^* where $r^*u + h$ is tangent to u^3 and also intersects it at a second (distinct)

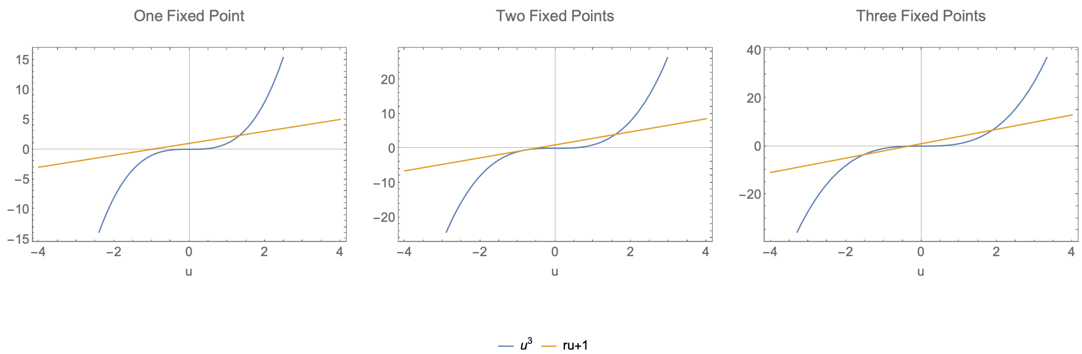


FIGURE 9.2. For fixed h (here $h = 1$) as r increases toward $r^* = 3\left(4^{-\frac{1}{3}}\right)$ we see one fixed point bifurcates into three fixed points.

location. For $r < r^*$, we have one fixed point. For $r > r^*$, we have three fixed points. We can find this point using a first order Taylor expansion of u^3 about a point $u = u_0$. We would need:

$$u_0^3 + 3u_0^2(u - u_0) = h + ru$$

The expression on the left is the tangent line while the expression on the right is the line we are given. Equating terms with and without u we have:

$$u_0^3 - 3u_0^3 = -2u_0^3 = h \tag{9.1}$$

$$3u_0^2 = r \tag{9.2}$$

Since h is fixed we have:

$$u_0 = -\sqrt[3]{\frac{h}{2}}$$

$$r^* = 3\left(\sqrt[3]{\frac{h}{2}}\right)^2$$

It is worth noting that $r^* > 0$ and hence if $r < 0$ we can be certain there is at most 1 fixed point (we will use this later).

For each instance of this problem, the exact fixed points need to be found numerically and their stability determined from $f'(u) = r - 3u^2$. However, we can make an educated guess about their stability. The potential function $V(u)$ is a positive quartic. This suggests it should have two local minima and one local maxima. Thus we expect to see two stable equilibria and one unstable equilibrium between these stable points. This is what we see for the case when $h = 1$ and $r = 3$ in Fig. 9.3. The asymmetric bifurcation diagram is illustrated in Fig. 9.4. Notice it resembles the pitchfork bifurcation we have seen, but the symmetry breaking term $h > 0$ causes the distortion.

The physical interpretation of this diagram is the more interesting part. For $r < r^*$, there is a single stable fixed point; that is there is one state of magnetization for the sample (silverware). As r increases past r^*, there are two stable magnetization equilibria and the system cannot move between them (because it would require passing through an unstable

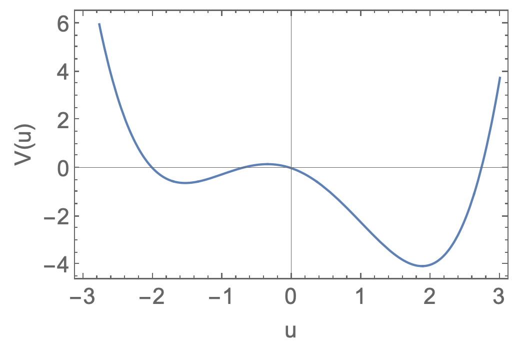


FIGURE 9.3. The potential function for $r = 3$, $h = 1$.

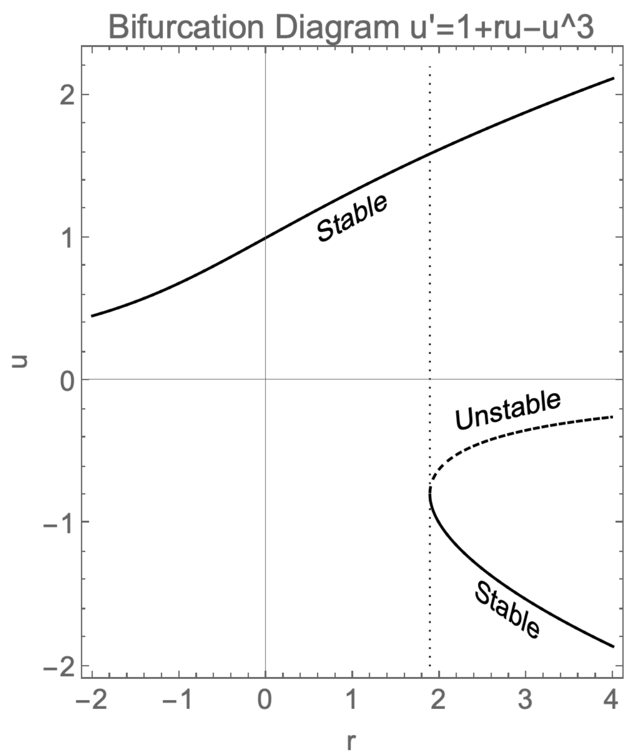


FIGURE 9.4. (Left) The imperfect pitchfork bifurcation exhibits symmetry breaking, but the expected one stable equilibrium becoming two stable and one unstable equilibria.

fixed point). These two stable equilibria correspond to the two possible magnetic regimes to which the system will stabilize depending on the initial condition.

Example 9.3 (Magnetizing a Metal - Part 2)**.** In the previous example, we varied r. Let us now vary h. We know already that for any h, if $r < 0$, there is only one fixed point (and it is stable). Now suppose $r > 0$. We can use Eqs. (9.1) and (9.2) again (assuming r is fixed) to see that:

$$u_0 = \pm\sqrt{\frac{r}{3}}$$
$$h^* = \pm 2\sqrt{\left(\frac{r}{3}\right)^3}.$$

That is, there are two critical values of h. The behavior (and emergence) of these two critical h values is shown in Fig. 9.5. This makes sense. Consider Fig. 9.2, where we see as h moves

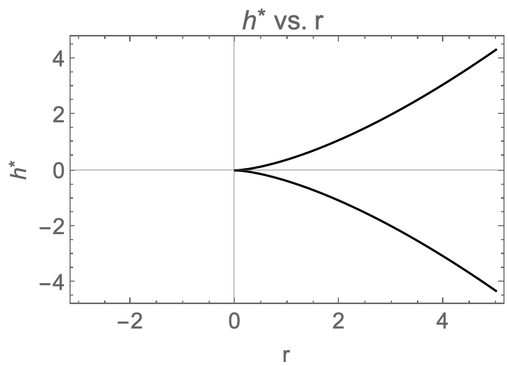


FIGURE 9.5. The two critical values of h emerge as r crosses from negative to positive.

up and down, the point of tangency changes on the curve; i.e., there are two potential tangent points depending on h. This is not the case when h is fixed and r changes. Thus, as we vary h, two fixed points emerge and then disappear. For simplicity, let:

$$h^* = 2\sqrt{\left(\frac{r}{3}\right)^3}.$$

If $h \in (-h^*, h^*)$, then there are three fixed points. Otherwise there is one fixed point (or two in the boundary cases – which we'll ignore). This is a type of *saddle-node* bifurcation in which two fixed points emerge (out of nowhere) with opposite stability. The bifurcation diagrams for the case when $r < 0$ (showing one stable fixed point) and $r > 0$ (showing regions where there are one, two or three fixed points) is shown in Fig. 9.6.

We can now construct a physical interpretation. Since h is a function of H the applied magnetic field varying h means varying the magnetic field. As we change the magnetic field, the resulting magnetization (fixed point) changes from one that is *not dependent* on the initial condition of the object (one stable fixed point) to one that is dependent on the initial condition of the sample (two stable fixed points separated by an unstable fixed point). Here, "initial condition" means how magnetized the object is at the start of time. Thus, objects that are highly magnetized tend to stay magnetized. However, if the magnetic field is strong enough, the long-term magnetization again no longer depends on the initial condition. When $h \in (-h^*, h^*)$ the object (silverware) is said to exhibit *hysteresis* – that is a a dependence on its history (starting condition).

Remark 9.4. There are many other bifurcations in the literature and we don't have time to deal with them all. Bifurcation theory forms an important part of the study of nonlinear

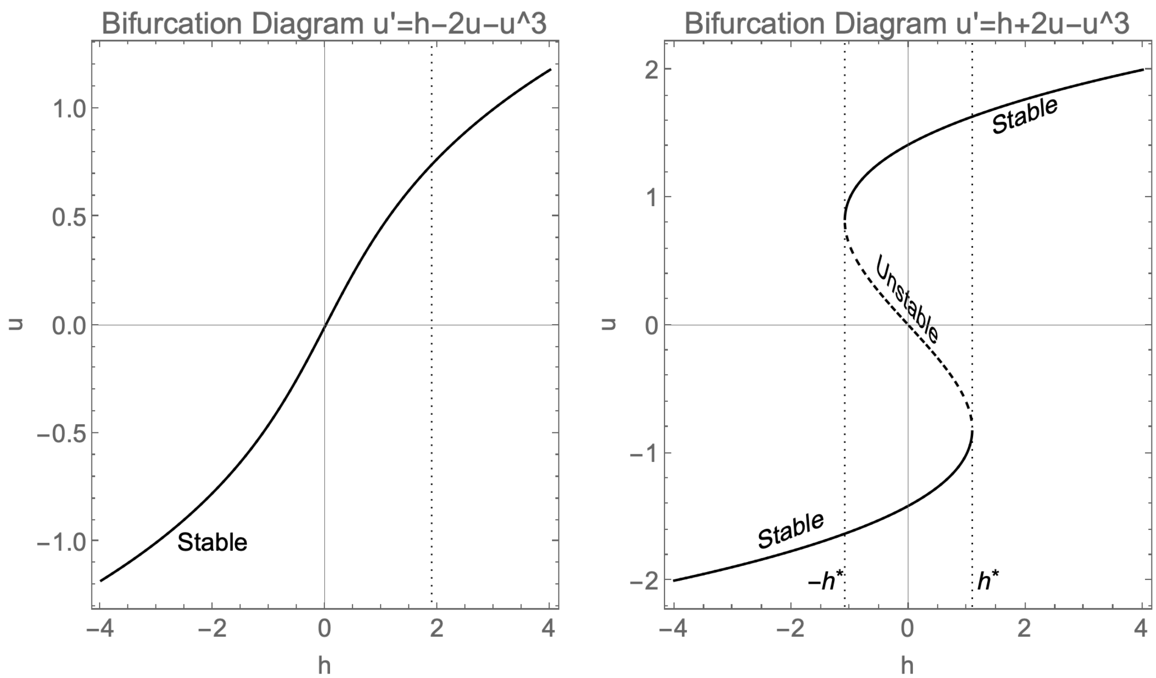


FIGURE 9.6. (Left) The bifurcation diagram for u^* vs. h when $r < 0$ shows there is only one stable equilibrium. (Right) The bifurcation diagram for u^* vs. h when $r > 0$ shows there is are three equilibria when $h \in (-h^*, h^*)$, two stable and one unstable. When h is outside this interval, there is only one stable equilibrium.

dynamics – which we are preparing to leave for a while in order to return to the problem of finding explicit solutions for ordinary differential equations.

Module 3

Quantitative Analysis of First Order ODEs

LESSON 10

1. Linear First Order Equations - Integrating Factors

Remark 10.1. Consider a linear first order ODE with form:

$$a(x)y' + b(x)y + c(x) = 0.$$

This equation can always be rewritten as:

$$y' + \frac{b(x)}{a(x)}y = -\frac{c(x)}{a(x)}$$

or more simply:

$$y' + p(x)y = q(x)$$

for functions $p(x)$ and $q(x)$. We will assume all ODE's are in this form for the rest of this section.

Derivation 10.2. Consider the ODE:

$$y' + p(x)y = q(x)$$

and let $\mu(x)$ be a function to be determined. We will call this an integrating factor. Then we can write:

$$\mu(x)y' + [\mu(x)p(x)]\,y = q(x)\mu(x)$$

If we could force:

$$\mu'(x) = \mu(x)p(x), \tag{10.1}$$

then we would have:

$$\mu(x)y' + [\mu(x)p(x)] = \mu y' + \mu' y = \frac{d}{dx}(\mu y) = q(x)\mu(x).$$

Using the Fundamental Theorem of Calculus we could then integrate to see:

$$\mu y = \int \mu(x)q(x)\,dx.$$

Then we would have a solution:

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)\,dx\right),$$

where we would introduce a constant of integration after integration on the right-hand-side. All that remains is to make Eq. (10.1) true is to find the missing $\mu(x)$.

Recall from the chain rule for some arbitrary differentiable function $P(x)$ we have:

$$\frac{d}{dx}\{\exp[P(x)]\} = P'(x)\exp[P(x)].$$

Therefore, if $P(x)$ is the antiderivative (with no arbitrary constant) of $p(x)$, then clearly it is differentiable and we would have:

$$\frac{d}{dx}\left\{\exp[P(x)]\right\} = p(x)\exp[P(x)].$$

Therefore, we could define $\mu(x) = \exp[P(x)]$ to make Eq. (10.1) true. Written in integral form, we have:

$$\mu(x) = \exp\left[\int p(x)\,dx\right].$$

Again, we **do not** introduce a constant of integration into $\mu(x)$. Therefore we have deduced that the solution to the ODE is:

$$y(x) = \frac{1}{\exp\left[\int p(x)\,dx\right]}\left(\int \exp\left[\int p(x)\,dx\right] q(x)\,dx\right).$$

Remark 10.3. The function $\mu(x)$ is called an *integrating factor*. It is a function that makes it possible to explicitly solve (integrate) the ordinary differential equation.

Remark 10.4. As a consequence of the derivation, we have proved a theorem.

THEOREM 10.5. *If $p(x)$ and $q(x)$ are continuous on the interval $I = (a, b)$ and $x_0 \in I$, then the initial value problem:*

$$y' + p(x)y = q(x) \quad y(x_0) = y_0$$

has a unique solution. Furthermore, if $P(x)$ is the antiderivative (with no arbitrary constant) of $p(x)$ and:

$$\mu(x) = \exp\left[P(x)\right],$$

then the unique solution is given by:

$$y(x) = \frac{1}{\mu(x)}\left(H(x) + C\right),$$

where $H(x)$ is the antiderivative (with no arbitrary constant) of the function $h(x) = \mu(x)q(x)$ and:

$$C = \mu(x_0)y_0 - H(x_0).$$

□

Example 10.6. Consider the initial value problem:

$$y' + \cos(x)y = \cos(x) \quad y(0) = 0.$$

Here we have: $p(x) = q(x) = \cos(x)$. Integrating we see $P(x) = \sin(x)$ and so:

$$\mu(x) = \exp[\sin(x)].$$

Then $h(x) = \cos(x)\exp[\sin(x)]$. Integrating we have:

$$H(x) = \int \cos(x)\exp[\sin(x)]\,dx.$$

Let $u = \sin(x)$ and $du = \cos(x)\,dx$. Using u substitution the integral becomes:

$$H(x) = \int e^u\,du = e^u = \exp[\sin(x)].$$

The solution general solution is:

$$y(x) = \frac{1}{\exp[\sin(x)]}\left(\exp[\sin(x)] + C\right) = 1 + C\exp[-\sin(x)].$$

Evaluating at $x_0 = 0$, $y_0 = 0$ gives a constant:

$$C = \exp[\sin(0)] \cdot (0) - \exp[\sin(0)] = -1.$$

The specific solution is then:

$$y(x) = 1 - \exp[-\sin(x)].$$

We can check this result. Note:

$$\frac{d}{dx}\left\{1 - \exp[-\sin(x)]\right\} = \exp[-\sin(x)]\cos(x)$$

So:

$$y' + \cos(x)y = \exp[-\sin(x)]\cos(x) + \cos(x)\left\{1 - \exp[-\sin(x)]\right\} = \cos(x),$$

as required.

1. Constants of Integration

Remark 11.1 (Constants of Integration)**.** You'll notice we've had some tortured language in dealing with those constants of integration. There is a nice mathematical way around this when given an initial condition. For example, if you have a separable ODE:

$$f(y)\frac{dy}{dx} = g(x) \quad y(x_0) = y_0$$

you write:

$$\int_{y_0}^{y} f(s)\,ds = \int_{x_0}^{x} g(t)\,dt.$$

Here s and t are both dummy[1] variables.

Example 11.2. To be more concrete if we had $\dot{x} = \alpha x$ then we have:

$$\int_{x_0}^{x} \frac{ds}{s} = \int_{t_0}^{t} dt.$$

This implies:

$$\log(x) - \log(x_0) = \log\left(\frac{x}{x_0}\right) = \alpha(t - t_0).$$

Simplified this yields the familiar answer:

$$x(t) = x_0 \exp[\alpha(t - t_0)],$$

which is identical to the solution we've seen when $t_0 = 0$.

Derivation 11.3. To see how this works in the context of Derivation 10.2 , let's go back to this step:

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = q(x)\mu(x).$$

We can now write this as:

$$d\left[\mu(x)y(x)\right] = q(x)\mu(x)\,dx,$$

and integrate:

$$\int_{x_0}^{x} d\left[\mu(s)y(s)\right] = \int_{x_0}^{x} q(s)\mu(s)\,ds, \tag{11.1}$$

[1] British sense, not North American: "Something designed to serve as a substitute."

The expression on the left-hand-side is called a Riemann–Stieltjes integral (or sometimes just a Stieltjes integral). Following our previous work, let:

$$\int_{x_0}^{x} q(s)\mu(s)\,ds = H(x) - H(x_0),$$

where:

$$H(x) = \int \mu(x)q(x)\,dx,$$

which now represents the formal antiderivative (no constant of integration). Then Eq. (11.1) simplifies to:

$$\mu(x)y(x) - \mu(x_0)y(x_0) = H(x) - H(x_0).$$

Note that $y(x_0) = y_0$ by assumption. Solving for $y(x)$ we have:

$$y(x) = \frac{1}{\mu(x)}\left[H(x) + \mu(x_0)y_0 - H(x_0)\right],$$

just as before. You can decide for yourself which of these two methods of derivation is less painful.

2. Decay Paths and Cascading Linear Systems

Remark 11.4. We can use our results in general solutions to first order linear systems to analyze nuclear decay paths. The idea for these examples comes from [**BC98**].

Example 11.5. Uranium-234 (^{234}U) decays into Thorium-230 (^{230}Th), which in turn decays into various other elements (Radium, Neon and Mercury). The half-life of ^{234}U is $\tau_1 = 2.46 \times 10^5$ years while the half-life of ^{230}Th is $\tau_2 = 7.54 \times 10^4$ years. Let k_1 be the decay constant of ^{234}U and k_2 be the decay constant of ^{234}U. Recall from Eq. (2.1) that:

$$k_i = \frac{\log(2)}{\tau_i},\; i = 1, 2.$$

If y_1 is the quantity of ^{234}U and y_2 is the quantity of ^{230}Th, then we can model the two quantities as:

$$\begin{aligned} \dot{y}_1 &= -k_1 y_1 \\ \dot{y}_2 &= k_1 y_1 - k_2 y_2. \end{aligned}$$

If $y_1(0) = m_1$ and $y_2(0) = m_2$ are the initial masses, we can compute:

$$y_1(t) = m_1 \exp(-k_1 t),$$

which yields an ODE for y_2:

$$\dot{y}_2 + k_2 y_2 = k_1 m_1 \exp(-k_1 t).$$

We can now compute the integrating factor:

$$\mu(t) = \exp\left(\int k_2\,dt\right) = e^{k_2 t}.$$

Then we have:

$$h(t) = e^{k_2 t} \cdot k_1 m_1 e^{-k_1 t} = k_1 m_1 e^{(k_2 - k_1)t}.$$

Integrating we see:

$$H(t) = \int k_1 m_1 e^{(k_2-k_1)t}\,dt = \frac{k_1 m_1}{k_2 - k_1} e^{(k_2-k_1)t}.$$

Thus we have:

$$y_2(t) = e^{-k_2 t}\left(\frac{k_1 m_1}{k_2 - k_1} e^{(k_2-k_1)t} + C\right).$$

Solving for C we have:

$$C = m_2 - \frac{k_1 m_1}{k_2 - k_1}.$$

Thus the complete solution is:

$$y_2(t) = \frac{k_1 m_1}{k_2 - k_1} e^{-k_1 t} + m_2 e^{-k_2 t} - \frac{k_1 m_1}{k_2 - k_1} e^{-k_2 t} = \frac{k_1 m_1}{k_2 - k_1}\left(e^{-k_1 t} - e^{-k_2 t}\right) + m_2 e^{-k_2 t}.$$

Example 11.6. You can perform the steps we used in Derivation 10.2 on a specific example as an alternative to using the formula in the theorem. Consider the ODE:

$$\frac{1}{x} y' + \frac{2}{x^2} y = \frac{\sec^2(x)}{x^3}.$$

To solve this ODE, we first multiply through by x to obtain:

$$y' + \frac{2}{x} y = \frac{\sec^2(x)}{x^2}.$$

The integrating factor is:

$$\mu(x) = \exp\left(\int \frac{2}{x}\,dx\right) = x^2$$

Then:

$$x^2 \frac{dy}{dx} + 2xy = \frac{d}{dx}\left(x^2 y\right) = \sec^2(x).$$

Integrating we have:

$$x^2 y = \tan(x) + C.$$

We conclude that:

$$y(x) = \frac{C}{x^2} + \frac{\tan(x)}{x^2}.$$

As expected from the initial form of the ODE, the solution is only valid for initial conditions intervals where $\sec(x)$ and $1/x$ are continuous a fact that is reflected in the solution as well.

3. Exact Equations*

Derivation 11.7. Consider a nonlinear ODE with form:

$$\frac{dy}{dx} = -u(x,y)/v(x,y).$$

This equation is not necessarily separable, but we could write it as:

$$u(x,y) + v(x,y)\frac{dy}{dx}dy = 0. \tag{11.2}$$

We know that $y(x)$ is a function of x and we recall that if we have the implicitly define $\psi(x,y) = C$ (for some $C \in \mathbb{R}$) then the implicit derivative is computed as:

$$\frac{d}{dx}\psi(x,y) = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0,$$

which is really just a special application of the chain rule. Therefore *if there exists* an $\psi(x,y)$ so that:

$$\frac{\partial\psi}{\partial x} = u(x,y) \tag{11.3}$$

$$\frac{\partial\psi}{\partial y} = v(x,y), \tag{11.4}$$

then we know that a $y(x)$ must be defined implicitly by the equation $\psi(x,y) = C$ for some C, where C is the constant of integration and this is the solution to the differential equation. Notice we are making liberal use of the implicit function theorem here.

It is worth noting that Eq. (11.2) can be rewritten as:

$$u(x,y)\,dx + v(x,y)\,dy = 0, \tag{11.5}$$

where $u(x,y)\,dx + v(x,y)\,dy$ is called a *differential form*. If a function ψ exists so that Eqs. (11.3) and (11.4) hold, then ψ is called a *potential function*. In this case, since we have:

$$\frac{d}{dx}\psi(x,y) = u(x,y) + v(x,y)\frac{dy}{dx} = 0,$$

then we can write:

$$d\psi = u(x,y)\,dx + v(x,y)\,dy,$$

where $d\psi$ is called the *total differential of* ψ.

Example 11.8. Rather than working backwards to ψ, let's work forwards from ψ to a differential equation. Define:

$$\psi(x,y) = x^2 + y^2.$$

If $\psi(x,y) = r^2$ for $r \in \mathbb{R}$, then this is the equation for a circle or radius r. Differentiating we have:

$$\frac{d\psi}{dx} = 2x + 2yy' = 0.$$

This is a differential equation, which we can rewrite as:

$$y' = -\frac{x}{y}.$$

Obviously this is a separable differential equation, and easily solved, but in this case we know already the solution is given by the implicit function $\psi(x,y) = x^2 + y^2 = C$.

Definition 11.9 (Differential Form & Exact Equations)**.** A differential form $u(x,y)\,dx + v(x,y)\,dy$ is *exact* in a rectangle $R = [a,b] \times [c,d]$ if there is a potential function $\psi(x,y)$ such that:

$$\frac{\partial \psi}{\partial x} = u(x,y) \qquad \frac{\partial \psi}{\partial y} = v(x,y).$$

In this case $d\psi = u(x,y)\,dx + v(x,y)\,dy$. The resulting equation: $u(x,y)\,dx + v(x,y)\,dy = 0$ is called an *exact equation*.

Remark 11.10. We now want to develop some criteria that will help us determine whether a differential equation is exact.

Lemma 11.11. *Let $\psi(x,y)$ be differentiable for all $(x,y) \in [a,b] \times [c,d] = R$ (a rectangle). If:*

$$\frac{\partial \psi}{\partial x} = u(x,y) \qquad \frac{\partial \psi}{\partial y} = v(x,y).$$

then:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{11.6}$$

in R. That is, if $u(x,y)\,dx + v(x,y)\,dy = 0$ is exact, then Eq. (11.6) holds.

PROOF. We have:

$$\frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial v}{\partial x},$$

because the differential operator commutes. $\square$

LESSON 12

1. More on Exact Equations*

Remark 12.1. In Lemma Lemma 11.11 we found a necessary criterion for exactness. We now show this criterion is sufficient.

Lemma 12.2. *Let $u(x,y)$ and $v(x,y)$ be continuously differentiable functions in a rectangle R. If*

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},$$

then there is a differentiable potential function $\psi(x,y)$ defined in R such that

$$\frac{\partial \psi}{\partial x} = u(x,y) \qquad \frac{\partial \psi}{\partial y} = v(x,y).$$

Proof. We will construct $\psi(x,y)$. First for some (arbitrary) x_0 define:

$$\psi(x,y) = \int u(x,y)\,dx + C(y). \tag{12.1}$$

Here:

$$\int u(x,y)\,dx$$

is the formal antiderivative of $u(x,y)$ in terms of x. Notice in Eq. (12.1), since we are (essentially) integrating with respect to x, the constant of integration is not just a constant value but is a function of y. (To see this, imagine what would happen if the integral were definite. We would be left with an extra function of y.) As a result $\partial_x \Psi(x,y) = u(x,y)$.

We now wish to find a particular $C(y)$ so that $\partial_y \psi = v(x,y)$. This means we require:

$$\frac{\partial \psi}{\partial y} = v(x,y) = \frac{\partial}{\partial y}\left[\int u(x,y)\,dx + C(y)\right] = \frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right] + C'(y).$$

Solving for $C'(y)$ yields:

$$C'(y) = v(x,y) - \frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right]. \tag{12.2}$$

This equation is sensible (valid) if (and only if) the right-hand-side is not a function of x. This can be checked by differentiating with respect to x to see:

$$\frac{\partial}{\partial x}\left\{v(x,y) - \frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right]\right\} = \frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right].$$

Apply the Leibniz rule (because all functions are continuously differentiable) we can exchange integration and differentiation to see:

$$\frac{\partial}{\partial x}\left\{v(x,y) - \frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right]\right\} = \frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\left[\int \frac{\partial u}{\partial y}\,dx\right] = \\ \frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\left[\int \frac{\partial v}{\partial x}\,dx\right],$$

because $\partial_y u = \partial_x v$. But then we have:

$$\frac{\partial}{\partial x}\left\{v(x,y) - \frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right]\right\} = \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} = 0.$$

Thus $C'(y)$ is a function of y alone and the right-hand-side of Eq. (12.2) can be integrated in terms of y to produce a function. In particular, integrating Eq. (12.2) with respect to y yields:

$$C(y) = \int \left\{v(x,y) - \frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right]\right\} dy$$

and:

$$\psi(x,y) = \int u(x,y)\,dx + C(y).$$

This function is a potential function by construction. □

Remark 12.3. Putting Derivation 11.7 and Lemmas 11.11 and 12.2 together, we have proved a theorem.

THEOREM 12.4. *If $u(x,y)$ and $v(x,y)$ are continuously differentiable in a rectangle R, then the differential form $u(x,y)\,dx + v(x,y)\,dy$ is exact if and only if*

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

and therefore there is a potential function ψ with: $u = \partial_x\psi$ and $v = \partial_y\psi$. □

Example 12.5. This was all pretty abstract. Let's work an example and build some clear steps to determine whether a differential equation is exact. Consider the ODE:

$$\frac{dy}{dx} = -\frac{3x^2 + 2xy + 2}{x^2 + 6y^2}.$$

Step 1: Write the equation as a differential form: We have:

$$(3x^2 + 2xy + 2)\,dx + (x^2 + 6y^2)\,dy = 0.$$

Step 2: Find $u(x,y)$ and $v(x,y)$ and check if $\partial_y u = \partial_x v$: We have:

$$u(x,y) = 3x^2 + 2xy + 2 \qquad v(x,y) = x^2 + 6y^2.$$

We check:

$$\frac{\partial u}{\partial y} = 2x = \frac{\partial v}{\partial x}.$$

Since the criterion we established is met, we have an exact ODE.

Step 3: Compute $v(x,y) - \partial_y \int u(x,y)\,dx$: We can compute $C'(y)$ as:

$$C'(y) = v(x,y) - \frac{\partial}{\partial y}\left[\int u(x,y)\,dx\right] = (x^2+6y^2) - \frac{\partial}{\partial y}\left[\int (3x^2+2xy+2)\,dx\right] =$$
$$(x^2+6y^2) - \frac{\partial}{\partial y}\left[x^3+x^2y+2x\right] = (x^2+6y^2) - x^2 = 6y^2.$$

As expected $C(y)$ is just a function of y.

Step 4: Compute $C(y)$ **by integration**: This step is easy:

$$C(y) = \int 6y^2\,dy = 2y^3.$$

Step 5: Compute $\psi(x,y)$ **as** $\int u(x,y)\,dx + C(y)$**:** We have:

$$\psi(x,y) = \int u(x,y)\,dx + C(y) = \int (3x^2+2xy+2)\,dx + 2y^3 = x^3+x^2y+2x+2y^3.$$

Step 6: Solve the ODE: The solution is:

$$\psi(x,y) = x^3+x^2y+2x+2y^3 = A,$$

for some constant A that can be determined by initial conditions, if they are given. The direction field of the ODE and a contour plot of $\psi(x,y)$ are shown illustrating that $\psi(x,y) = A$ is the general solution to the ODE.

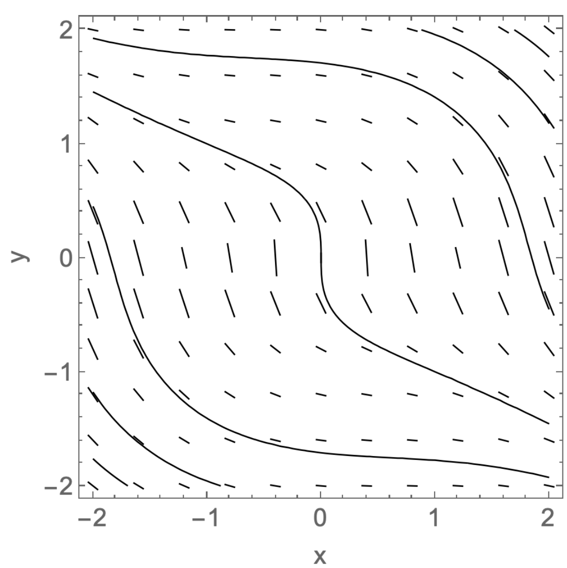


FIGURE 12.1. The contour plot of $\psi(x,y)$ is shown along with the direction field of the ODE.

Example 12.6 (Flow Around a Cylinder)**.** This next example is algebraically messy, so use a computer algebra system to work out the details yourself, if you want. The details of fluid flow can be found in [BB00]. Consider the unusual exact differential form:

$$\frac{2R^2Uxy}{\left(x^2+y^2\right)^2}\,dx + \left[\frac{R^2U\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} + U\right]\,dy = 0,$$

where R and U are two positive constants. We can verify (with a computer algebra system) that this yields an exact ODE by showing:

$$\frac{\partial}{\partial y}\left[\frac{2R^2Uxy}{\left(x^2+y^2\right)^2}\right] = \frac{2R^2Ux\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3} = \frac{\partial}{\partial x}\left[\frac{R^2U\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} + U\right].$$

Therefore, this defines and exact differential equation. Amazingly, one can show that:

$$C'(y) = U + \frac{R^2U\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} - \frac{\partial}{\partial y}\left[\int \frac{2R^2Uxy}{\left(x^2+y^2\right)^2}\,dx\right] = U.$$

Therefore, $C(y) = Uy$. Then:

$$\psi(x,y) = Uy + \int \frac{2R^2Uxy}{\left(x^2+y^2\right)^2}\,dx = Uy\left(1 - \frac{R^2}{x^2+y^2}\right).$$

Solutions have the form:

$$Uy\left(1 - \frac{R^2}{x^2+y^2}\right) = A.$$

The question is: What is this? A decorated direction field plot will help. If $U = 1 = R$ we have the direction field shown below. Solutions to this ODE describe the streamlines around

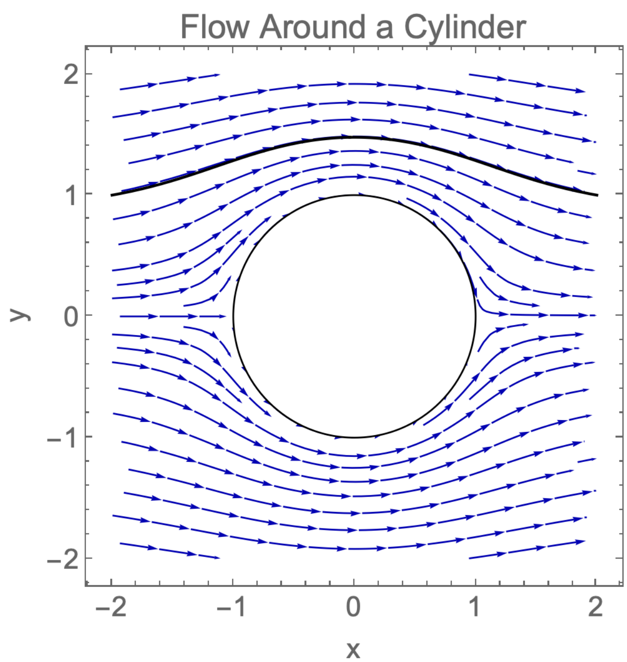


FIGURE 12.2. The solution to this ODE yields flow lines around a cylinder in an incompressible and irrotational fluid. The cylinder is shown from the overhead view.

a fluid that is incompressible and irrotational with a left-to-right velocity of U far away from the cylinder of radius R (shown here as a white circle).

Remark 12.7. Normally, getting to $\psi(x, y)$ in the last example would not be done by an ODE, but would be done by solving a PDE and manipulating that solution to derive the stream function directly. In particular, this explains (physically) why stream functions satisfy the equation $\psi(x, y) = A$. Also, the solution would normally be given in polar coordinates. I've short-circuited the derivation a little to give you a sense of where these types of results arise. Analysis of the flow around objects forms the basis for modern aerospace engineering.

1. Integrating Factors - Redux*

Derivation 13.1. Consider an equation we've already solved:

$$\frac{dy}{dx} + p(x)y = q(x).$$

Rewriting this we have:

$$[p(x)y - q(x)]\, dx + dy = 0. \tag{13.1}$$

Let $u(x, y) = p(x)y - q(x)$ and $v(x, y) = 1$. Clearly this equation is not exact unless $p(x) = 0$ because:

$$\frac{\partial u}{\partial y} = p(x) \neq \frac{\partial v}{\partial x} = 0.$$

However, recall we defined:

$$\mu(x) = \exp\left[\int p(x)\, dx\right].$$

Now observe we can multiply Eq. (13.1) by $\mu(x)$:

$$[p(x)y - q(x)]\, \mu(x)\, dx + \mu(x)dy = 0.$$

Solutions to this equation must (necessarily) be identical to the original equation but, this new equation is exact as we'll show. We have now have:

$$u(x, y) = p(x)y \exp\left[\int p(x)\, dx\right] - q(x) \exp\left[\int p(x)\, dx\right]$$

$$v(x, y) = \exp\left[\int p(x)\, dx\right].$$

We now see that:

$$\frac{\partial u}{\partial y} = p(x)\left[\int p(x)\, dx\right] = \frac{\partial v}{\partial x}.$$

Thus we have converted the original equation into an exact equation, which we know can be solved with a potential function. This idea can be generalized.

Definition 13.2 (Integrating Factor)**.** Consider the differential form:

$$f(x, y)\, dx + g(x, y)\, dy = 0.$$

If this does not yield an exact differential equation but there is a function $\mu(x, y)$ such that:

$$\mu(x, y)f(x, y)\, dx + \mu(x, y)g(x, y)\, dy = 0.$$

does yield an exact differential equation, then $\mu(x, y)$ is called an *integrating factor*.

Example 13.3. It is *non-trivial* to find an integrating factor (if one even exists). However, here is an example. Consider:

$$3y^2 + 8x + 4xy\frac{dy}{dx} = 0.$$

Then we have:

$$u(x,y) = 3y^2 + 8x \qquad v(x,y) = 4xy.$$

We have:

$$\frac{\partial u}{\partial y} = 6y \neq \frac{\partial v}{\partial x} = 4y.$$

Let $\mu(x,y) = \sqrt{x}$. Then we redefine:

$$u(x,y) = (3y^2 + 8x)\sqrt{x} \qquad v(x,y) = 4xy\sqrt{x}.$$

Now:

$$\frac{\partial u}{\partial y} = 6y\sqrt{x} = \frac{\partial v}{\partial x}.$$

We could now find a potential function that will yield solution curves to the original problem.

Derivation 13.4. We derive a special case when we can find an integrating factor. Suppose

$$f(x,y)\,dx + g(x,y)\,dy = 0.$$

is not an exact differential equation. We seek a function $\mu(x,y)$ so that:

$$\mu(x,y)f(x,y)\,dx + \mu(x,y)g(x,y)\,dy = 0.$$

This implies

$$\frac{\partial}{\partial y}\left[\mu(x,y)f(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x,y)g(x,y)\right].$$

Expanding we have:

$$f\frac{\partial \mu}{\partial y} + \mu\frac{\partial f}{\partial y} = g\frac{\partial \mu}{\partial x} + \mu\frac{\partial g}{\partial x}.$$

Factoring this gives the partial differential equation:

$$\mu\left[\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right] = g\frac{\partial \mu}{\partial x} - f\frac{\partial \mu}{\partial y}.$$

This is not easy to solve unless $\mu(x,y) = \mu(x)$; that is, μ is just a function of x. In this case, our PDE becomes and ODE:

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{g}\left[\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right]$$

just in case the right-hand-side is a function of x alone. That is:

$$\frac{1}{g}\left[\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right] = p(x).$$

In this case, we can solve:

$$\frac{1}{\mu}\frac{d\mu}{dx} = p(x)$$

to see:

$$\mu(x) = \exp\left[\int p(x)\,dx\right].$$

THEOREM 13.5. *Consider the differential equation:*

$$f(x,y) + g(x,y)\frac{dy}{dx} = 0. \tag{13.2}$$

If

$$\frac{1}{g}\left[\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right]$$

is a function of x alone, denoted $p(x)$, then when:

$$\mu(x) = \exp\left[\int p(x)\,dx\right].$$

the differential equation:

$$f(x,y)\mu(x) + g(x,y)\mu(x)\frac{dy}{dx} = 0$$

is exact and can be solved with a potential function $\psi(x,y) = A$ for a some constant A. This is also a solution to Eq. (13.2). $\square$

Example 13.6. Consider the ODE:

$$3y^2 + 8x + 4xy\frac{dy}{dx} = 0.$$

We have:

$$f(x,y) = 3y^2 + 8x \qquad g(x,y) = 4xy.$$

Then:

$$\frac{1}{g}\left[\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right] = \frac{1}{4xy}(6y - 4y) = \frac{1}{2x}.$$

Setting $p(x) = (2x)^{-1}$ we have:

$$\mu(x) = \exp\left[\int \frac{1}{2x}\,dx\right] = \sqrt{x}.$$

This agrees with Example 13.3.

Remark 13.7. If this trick does not work, then there may still be an integrating function, but it's a non-trivial case and well outside the scope of the course.

Module 4

Other Applications of First Order ODEs

1. Concentration Problems

Remark 14.1 (Single Compartment Model)**.** A *single compartment* model is a model of the time-varying amount of some substance that flows into and out of a single compartment. This is illustrated in Fig. 14.1. The general model follows the rule:

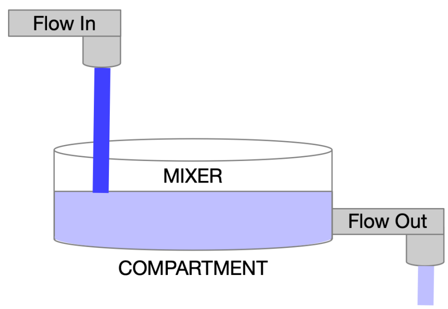


FIGURE 14.1. In a single compartment model a quantity of interest flows into and out of a compartment.

$$\frac{dQ}{dt} = r_{Q_{\text{in}}} - r_{Q_{\text{out}}}, \tag{14.1}$$

where $r_{Q_{\text{in}}}$ is the rate the quantity is flowing in and $r_{Q_{\text{out}}}$ is the rate the quantity is flowing out.

Example 14.2. Consider a tank containing a solution with volume 1 kl. The solution was a salt concentration of 35 g/l. Water with a salinity of 10 g/l is flowing into the tank at a rate of 10 l/min. The fully mixed brine solution is flowing out at the same rate. We can model the change in salinity (salt concentration) of the solution in the tank as a function of time.

Step 1 - Compute the volume as a function of time: We have:

$$\dot{V} = 10\,\text{l/min} - 10\,\text{l/min} = 0.$$

The units are in liters per minute. Therefore:

$$V(t) = V_0 = 1\,\text{kl} = 1000\,\text{l}.$$

Step 2 - Compute the mass of the salt as a function of time: Let $m(t)$ be the unknown mass of salt in the tank as a function of time. Using some dimensional analysis, we know the salt is flowing in according to the expression:

$$r_{m_{\text{in}}} = 10\,\text{l/min} \cdot 10\,\text{g/l} = 100\,\text{g/min}.$$

The rate salt is leaving is harder to compute. The concentration in the tank $m(t)/V(t)$ in units g/l. So we have:

$$r_{m_{\text{out}}} = 10\,\text{l/min} \cdot \frac{m(t)}{V(t)} = \frac{10m(t)}{1000} = \frac{1}{100}m(t).$$

Therefore we have:

$$\dot{m} = 100 - \frac{1}{100}m.$$

The initial mass can be computed as:

$$m_0 = 35\,\text{g/l} \cdot 1000\,\text{l} = 35000\,\text{g} = 35\,\text{kg}.$$

We can solve this equation any number of ways but in grams, the mass function is then:

$$m(t) = 5000\left[2 + 5\exp\left(-\frac{t}{100}\right)\right].$$

Step 3 - Compute the time-varying concentration: We know the concentration is $m(t)/V(t)$, so:

$$\rho(t) = \frac{5000\left[2 + 5\exp\left(-\frac{t}{100}\right)\right]}{1000} = 5\left[2 + 5\exp\left(-\frac{t}{100}\right)\right].$$

Example 14.3 (Time Varying Volume)**.** Suppose we start with the same problem but now the flow rate out is 8 l/min. We can follow the same steps but now $V(t)$ is not a constant.

Step 1 - Compute the volume as a function of time: We have:

$$\dot{V} = 10\,\text{l/min} - 8\,\text{l/min} = 2.$$

The units are in liters per minute. Therefore:

$$V(t) = V_0 + 2t = 1\,\text{kl} + 2t = 1000 + 2t.$$

Step 2 - Compute the mass of the salt as a function of time: As before we have:

$$r_{m_{\text{in}}} = 100\,\text{g/min}.$$

The rate salt is leaving is now different. As always, concentration in the tank $m(t)/V(t)$ in units g/l. So we have:

$$r_{m_{\text{out}}} = 8 \cdot \frac{m(t)}{V(t)} = \frac{8m}{1000 + 2t} = \frac{4m}{500 + t}.$$

Therefore we have:

$$\dot{m} = 100 - \frac{4m}{500 + t}.$$

This we can rewrite as:

$$\dot{m} + \frac{4}{500 + t}m = 100.$$

Using the integrating factor:

$$\mu(t) = \exp\left(\int \frac{4}{500+t}\,dt\right) = (t+500)^4$$

Then our formula is:

$$m(t) = \frac{1}{(t+500)^4}\left[\int 100(t+500)^4\,dt + C\right] = 20(t+500) + \frac{C}{(t+500)^4}.$$

We know from before that $t_0 = 0$ and $m(0) = 35000$ (grams). So the constant C is the very unwieldy:

$$C = 1,562,500,000,000,000 g \cdot \text{min} = 1.5625 \times 10^{15} g \cdot \text{min}.$$

As before, an expression for salt concentration is: $m(t)/V(t)$. We can plot the resulting concentration in Fig. 14.2:

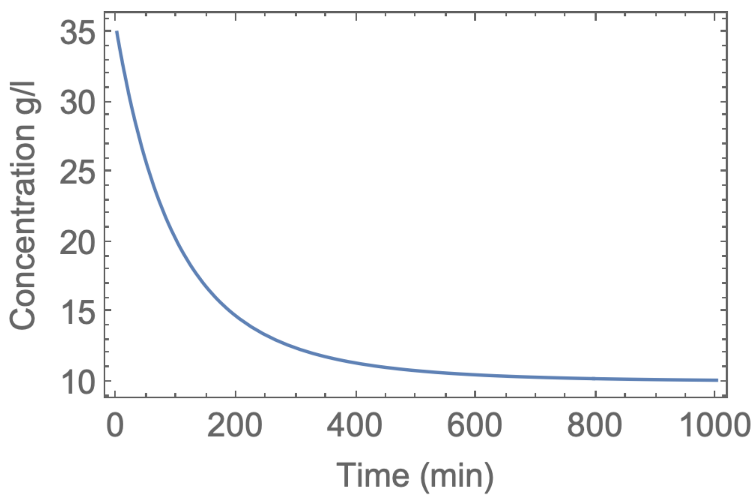


FIGURE 14.2. The concentration of salt decreases over time as a volume of fluid increases and the less concentrated brine flows is introduced into the system.

2. Newton's Law of Cooling

Remark 14.4. Compared to mixing problems, Newton's law of cooling is easy.

Definition 14.5 (Newton's Law of Cooling-Simplified Form)**.** Suppose a sample of substance has time varying temperature $u(t)$ and is placed in an environment with temperature u_{env}. Then:

$$\dot{u} = k(u_{\text{env}} - u),$$

where $k > 0$ is the coefficient of heat transfer and depends on the substance.

Remark 14.6. The more sophisticated form of Newton's law of cooling takes into account the surface area through which heat is being transferred and is expressed in terms of watts rather than degrees Kelvin (or celsius). The dynamics, however, are identical so for our purposes the simplified form is sufficient. The next theorem should be clear.

THEOREM 14.7. *Suppose a sample of substance has time varying temperature $u(t)$ with $u(0) = u_0$ and is placed in an environment with temperature u_{env} and coefficient of heat transfer k. Then:*

$$u(t) = u_{env} + (u_0 - u_{env})e^{-kt}.$$

□

Derivation 14.8 (Newton's Law of Cooling with Internal Heat)**.** Suppose the sample we are studying has an internal heating (or cooling) property. An example of this is a temperature controlled building, but there could be an internal endothermic or exothermic reaction occurring. In this case we can think of the net energy (heat) being transferred into the sample. This is another single compartment model with the sample being treated as the compartment and the energy as the substance being transferred. Furthermore, there is no reason to assume the environmental temperature is constant. The resulting differential equation is then:

$$\dot{u} = k(u_{\text{env}}(t) - u) + q(t),$$

where $u_{\text{env}}(t)$ is the time-varying environmental temperature and $q(t)$ is the time-varying internal heat of the sample. It's units must be in temperature per unit time, e.g. K/s. The function $q(t)$ is often called a *forcing function*.

This equation can be rewritten as:

$$\dot{u} + ku = ku_{\text{env}}(t) + q(t).$$

Consequently, we can solve it using the integrating factor methods we have already discussed.

Example 14.9. A sample is maintained in a laboratory at 0 C and is generating heat with a decaying forcing function $q(t) = 100e^{-t}$ meausred in degrees celsius per second. The sample's initial temperature is 100 C and the coefficient of heat transfer is 1. We can find the temperature of the sample by solving:

$$\dot{u} + u = 100e^{-t}.$$

Set:

$$\mu(t) = \exp\left[\int dt\right] = \exp(t).$$

Then:

$$u(t) = \frac{1}{\exp(t)}\left(\int 100e^{t}e^{-t}\,dt\right) = 100te^{-t} + Ce^{-t}$$

We know u(0) = 100, so $C = 100$. The solution is:

$$u(t) = (100 + 100t)\,e^{-t}.$$

Using Theorem 14.7, we can compare this to the case when there is no forcing function. Then the solution is:

$$\tilde{u}(t) = 100e^{-t}.$$

The internal heating of the object slows the natural cooling process (as one would expect). This is illustrated in the figure below.

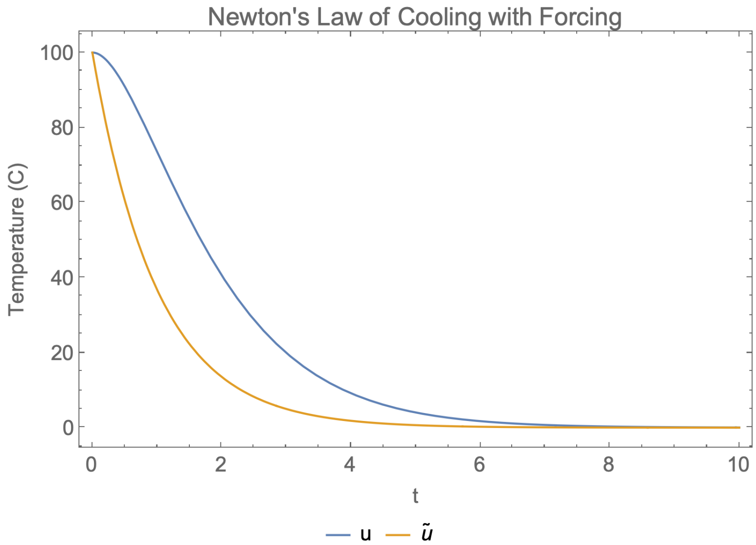


FIGURE 14.3. Newton's law of cooling with and without forcing is shown.

1. Circuits - Part 1

Definition 15.1 (Current)**.** *Current* is the flow of electric charge through a medium (wire). It is usually measured in *amperes*, which are *coulombs per second*. Thus, if q is the charge in the medium, then $\dot{q}$ is the current.

Definition 15.2 (Voltage)**.** *Voltage* is a measure of the amount of energy per unit charge. Specifically, it is the amount of work (in watts) that must be done per unit of charge to from one location to another. It it measured in watts per coulomb or volts.

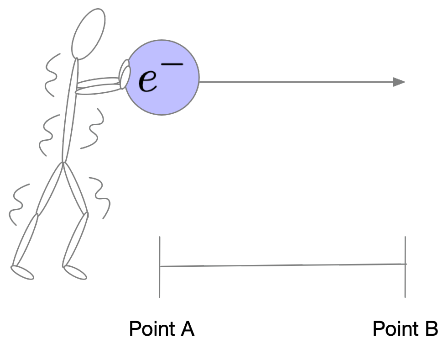


FIGURE 15.1. Voltage is a measure of work or energy per unit charge. Specifically the work per coloumb needed to move a test charge between two points.

Definition 15.3 (Resistor)**.** A *resistor* is an electrical device that resists the flow of current when placed in a circuit. Resistance is measured in *ohms*, which is a relatively messy derived unit.

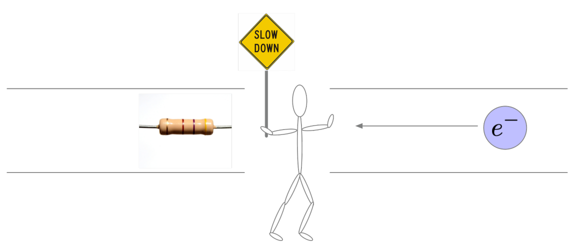


FIGURE 15.2. A resistor is an electrical device that opposes current flow thereby. Resistors cause voltages to decrease by converting some energy in the charge to heat.

Definition 15.4 (Capacitor)**.** A *capacitor* is an electrical device composed of two metal plates separated by a non-conducting material called a dialectric. A charged capacitor has opposite signed charges built up on the opposing metal plates allowing the charged to be stored. If the two plates are electrically connected, the charge imbalance will quickly come

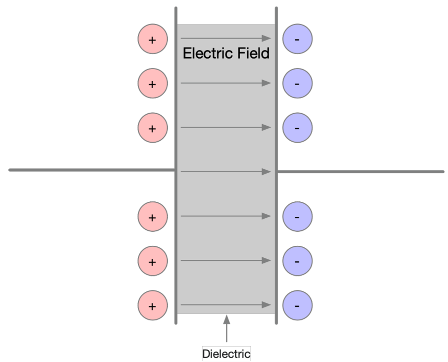


FIGURE 15.3. A capacitor stores charge on opposite, non-touching metal plates.

to equilibrium causing (sometimes) explosive effects. If a capacitor is charged with a battery of voltage V, then the amount of charge Q stored is:

$$Q = CV, \tag{15.1}$$

where C is a constant dependent on the capacitor (called the capacitor's *capacitance*).

Definition 15.5 (Ohm's Law)**.** Ohm's law is an empirical law relating voltage, resistance and current. In particular, if V is the voltage in a circuit, $\dot{Q}$ is the current and R is the circuit resistance, then:

$$V = R\dot{Q} = RI \tag{15.2}$$

Derivation 15.6. When the two plates of a capacitor are connected electrically through a resistor, the charge drains slowly from the capacitor. Details of this derivation are taken from [PM13].

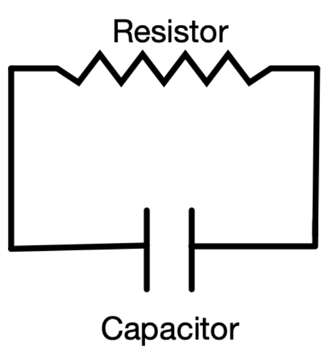


FIGURE 15.4. A circuit with a capacitor and resistor slowly drains charge away from the capacitor causing the voltage (energy) in the system drain way as heat.

We can analyze this situation. From Eq. (15.1) we have:

$$\dot{Q} = C\dot{V}.$$

Kirchoff's loop law states that the sum of all currents inside a circuit must be equal to zero. The two currents (for the resistor and capacitor) are:

$$I = \dot{Q} = \frac{V}{R},$$

and

$$I = \dot{Q} = C\dot{V}.$$

Therefore:

$$C\dot{V} + \frac{V}{R} = 0$$

or:

$$\dot{V} = -\frac{V}{RC}.$$

Consequently, the time-varying voltage in the circuit is:

$$V(t) = V_0 \exp\left(-\frac{t}{RC}\right).$$

This makes sense. As the energy slowly drains out of the circuit, the voltage falls to zero asymptotically.

2. Circuits - Part 2

Definition 15.7 (Inductor)**.** An *inductor* is a coil of wire that creates a magnetic field whenever a changing current (I) passes through it. Amazingly, the magnetic field induces current that flows in the opposite direction. Specifically, the voltage (electromotive force) is given by:

$$V(t) = V_0 - L\dot{I}, \tag{15.3}$$

where V_0 is a constant (initial) voltage and $\dot{I}$ is the rate of change of current and L is the inductance of the coil. This is an intrinsic property.

Derivation 15.8 (RL Circuit)**.** Details of this derivation can be found in [**PM13**]. Suppose we have an inductor and a resistor in a circuit. We know by combining Eqs. (15.2) and (15.3) that:

$$V(t) = V_0 - L\dot{I} = IR,$$

since $\dot{Q} = I$. That is:

$$\dot{I} + \frac{R}{L}I = V_0.$$

This equation can be easily solved to show that:

$$I(t) = \frac{V_0}{R}\left[1 - \exp\left(-\frac{R}{L}t\right)\right].$$

Here as $t \to \infty$, the current approaches a fixed point V_0/R asymptotically as the counter-force of the inductor drops overtime (because the current slowly stops changing).

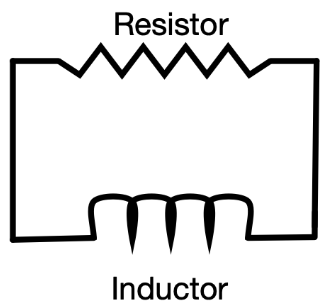


FIGURE 15.5. A circuit with an inductor and resistor allows the current to asymptotically approach a level expected from a specific voltage (electromotive force).

Remark 15.9. This derivation can be generalized by assuming V_0 is actually a function of time. The resulting ODE can be solved with an integrating factor. Also, we note we're being a little "fast-and-loose" with our use of voltage vs. electromotive force in the previous derivation. For a more complete discussion, take a course in electromagnetism.

Module 5

Second Order ODEs

LESSON 16

1. Complex Numbers & Preliminaries

Remark 16.1. You will notice that none of the solutions to the first order ODE's exhibited any oscillations. In fact, except when they blew up, they were fairly boring. However, from Example 2.15, where we solved the $y'' = -\alpha^2 y$, we know that solutions two second order ODE's can oscillate. Oscillations mean its time to introduce complex numbers.

Definition 16.2 (Imaginary and Complex Numbers)**.** Let $i = \sqrt{-1}$. A *complex number* is a number with form $a + bi$ for $a, b \in \mathbb{R}$. A complex number $a + bi$ is *pure imaginary* if $a = 0$. The set of complex numbers is denoted $\mathbb{C}$.

Remark 16.3. Each complex number $a + bi$ can be visualized as a point (a, b) in the plane. We have the notation $\text{Re}(z) = a$ and $\text{Im}(z) = b$ if $z = a + bi$.

Definition 16.4 (Complex Conjugate)**.** If $z = a+bi$ then its *complex conjugate* is $\bar{z} = a-bi$.

Remark 16.5 (Operations)**.** If $a + bi$ is a complex number and $r \in \mathbb{R}$, then:

$$r(a + bi) = ra + rbi \tag{16.1}$$

Complex numbers add and multiply by the following rules:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Let $z = a + bi$. Then $z\bar{z} = a^2 + b^2$, which is real. For complex number z, define $|z| = \sqrt{z\bar{z}}$. This makes sense it is the distance of the point (a, b) to the origin, thus generalizing the idea of absolute value. Finally, if $w, z \in \mathbb{C}$ and $z \neq 0$, we can define division:

$$\frac{w}{z} = \frac{w\bar{z}}{|z|^2}.$$

Now the denominator is real and can be divided into numerator using Eq. (16.1).

Proposition 16.6. *Consider the equation $p(s) = as^2 + bs + c = 0$.*

(1) *If $b^2 - 4ac > 0$, then there are two real numbers r_1 and r_2 satisfying $p(r_i) = 0$ ($i = 1, 2$).*
(2) *If $b^2 - 4ac = 0$, then there are one real numbers r satisfying $p(r) = 0$.*
(3) *If $b^2 - 4ac < 0$, then there is a complex number z such that $p(z) = p(\bar{z}) = 0$.*

□

Remark 16.7. The interesting part of the last proposition is that complex roots of $p(x)$ com in conjugate pairs. This is generally true of all polynomials and is a part of the fundamental theorem of algebra, which we will not cover.

2. Characteristic Equations

Remark 16.8. Recall we discussed linear operators in Definition 3.16. We will use them now.

Derivation 16.9. Consider the second order differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0. \tag{16.2}$$

We can write this as:

$$L(y) = 0,$$

with

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$$

Consider the action of L on $e^{\alpha x}$:

$$L\left(e^{\alpha x}\right) = a\frac{d^2}{dx^2}\left(e^{\alpha x}\right) + b\frac{d}{dx}\left(e^{\alpha x}\right) + ce^{\alpha x} = \\ a\alpha^2 e^{\alpha x} + b\alpha e^{\alpha x} + ce^{\alpha x} = (a\alpha^2 + b\alpha + c)e^{\alpha x}.$$

Observe that if $(as^2+bs+c)e^{\alpha x} = 0$ *for all* x, then we would require that α solves $as^2+bs+c = 0$. Thus, we have derived a general solution Eq. (16.2). In light of Proposition 16.6, we now have a few cases to consider because it is perfectly clear what this means when there are two real roots. When there are two complex roots, this is less clear and when there is only one real root, we expect another solution to be lurking around somewhere.

Definition 16.10 (Characteristic Equation)**.** Given the second order differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

the *characteristic equation* is $as^2 + bs + c = 0$.

THEOREM 16.11 (Two Real Roots). *Consider the second order differential equation:*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

If the characteristic equation $as^2 + bs + c = 0$ has two distinct roots $r_1, r_2 \in \mathbb{R}$, then the general solution to the ODE is:

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}.$$

PROOF. We know from Derivation 16.9 that both e^{r_1x} and e^{r_2x} must solve the ODE. The operator defining this ODE is linear and therefore by Theorem 2.14 (superposition) it follows that $C_1e^{r_1x} + C_2e^{r_2x}$ also solves the ODE for arbitrary constants C_1 and C_2. □

Example 16.12. To solve the ODE:

$$y'' + 3y' + 2y = 0$$

we solve $x^2 + 3x + 2 = 0$. Note $x^2 + 3x + 2 = (x + 1)(x + 2)$ so there are two roots $r_1 = -1$ and $r_2 = -2$. The solution to the ODE is:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

If we were given $y(0) = 2$ and $y'(0) = -3$ then we could solve for the two unknown constants of integration as:

$$\begin{aligned} &C_1 + C_2 = 2 \\ &-C_1 - 2C_2 = -3 \end{aligned}$$

Then $C_1 = 1 = C_2$ and the specific solution is:

$$y(x) = e^{-x} + e^{-2x}.$$

3. The Case of Two Complex Roots

Remark 16.13. The next theorem would be proved in a complex analysis class. We'll take it on faith.

Lemma 16.14. *The following Taylor series expansions about $z = 0$ are valid for all $z \in \mathbb{C}$:*

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ \cos(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \end{aligned}$$

$\square$

Lemma 16.15 (Euler's Theorem*)**.** *Let $z = a + bi$. Then:*

$$e^z = e^a e^{bi} = e^a \left(\cos(b) + i \sin(b)\right).$$

$\square$

Proof. It is sufficient to show $e^{bi} = \cos(b) + i \sin(b)$. Apply Lemma 16.14:

$$e^{bi} = 1 + bi + \frac{(bi)^2}{2} + \frac{(bi)^3}{3!} + \cdots = \underbrace{\left[1 + \frac{(bi)^2}{2} + \cdots\right]}_{\text{Even Powers}} + \underbrace{\left[bi + \frac{(bi)^3}{3!} + \cdots\right]}_{\text{Odd Powers}}.$$

Recall $i^{2n} = (-1)^n$ and $i^{2n+1} = i(-1)^n$. The we have:

$$e^{bi} = \underbrace{\left(1 - \frac{b^2}{2} + \cdots\right)}_{\text{Even Powers}} + i \underbrace{\left(b - \frac{b^3}{3!} + \cdots\right)}_{\text{Odd Powers}} =$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{b^{2n}}{(2n)!} + i \left[\sum_{n=0}^{\infty} (-1)^n \frac{b^{2n+1}}{(2n+1)!}\right] = \cos(b) + i \sin(b).$$

$\square$

Derivation 16.16. Let $z = a+bi$ and $\bar{z} = a-bi$ be two conjugate roots of the characteristic equation $as^2 + bs + c = 0$ corresponding to the ODE:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

The immediate solution is $y(x) = K_1e^{(a+bi)x} + K_2e^{(a-bi)x}$, where K_1 and K_2 are now two arbitrary complex constants. Let $K_1 = A + Bi$ and $K_2 = C + Di$. Then we can write:

$$y(x) = (A + Bi)e^{ax}\left[\cos(bx) + i\sin(bx)\right] + (C + Di)e^{ax}\left[\cos(-bx) + i\sin(-bx)\right] = \\ (A + Bi)e^{ax}\left[\cos(bx) + i\sin(bx)\right] + (C + Di)e^{ax}\left[\cos(bx) - i\sin(bx)\right],$$

because $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. This solution is not satisfying because we want a *real* solution for our problem assuming we have real initial conditions. We can fix this by grouping the real and imaginary terms together. We have:

$$y(x) = Ae^{ax}\cos(bx) + Ce^{ax}\cos(bx) - Be^{ax}\sin(bx) + De^{ax}\sin(bx)+ \\ i\left[Ae^{ax}\sin(bx) - Ce^{ax}\sin(bx) + Be^{ax}\cos(bx) + De^{ax}\cos(bx)\right] = \\ e^{ax}\left[(A + C)\cos(bx) + (D - B)\sin(bx)\right] + ie^{ax}\left[(A - C)\sin(bx) + (B + D)\cos(bx)\right]$$

Remember A, B, C, and D are arbitrary so set $A = C$ and $B = -D$ and the imaginary part goes to zero leaving a solution:

$$y(x) = 2Ae^{ax}\cos(bx) + 2De^{ax}\sin(bx) = e^{ax}\left[C_1\cos(bx) + C_2\sin(bx)\right].$$

This is the general solution when there are two complex roots to the characteristic equation.

Theorem 16.17 (Two Complex Roots). *Consider the second order differential equation:*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

If the characteristic equation $as^2 + bs + c = 0$ has two complex conjugate roots $a + bi$ and $a - b_i$ with $a, b \in \mathbb{R}$, then the general solution to the ODE is:

$$y(x) = e^{ax}\left[C_1\cos(bx) + C_2\sin(bx)\right].$$

□

Example 16.18. Consider the ODE:

$$y'' - 2y' + 5y = 0.$$

The characteristic equation is: $x^2 - 2x + 5 = 0$ and it has roots:

$$x = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm 2i.$$

Then we have the solution:

$$y(x) = e^{x}\left[C_1\cos(2x) + C_2\sin(2x)\right].$$

If we are given the initial data $y(0) = 1$ and $y'(0) = -1$, then we would have:

$$y(0) = e^0\left[C_1\cos(0) + C_2\sin(0)\right] = C_1 = 1$$

and

$$y'(0) = e^0\left[\cos(0) + C_2\sin(0)\right] + e^0\left[-2\sin(0) + 2C_2\cos(0)\right] = 1 + 2C_2 = -1.$$

Therefore $C_2 = -1$. The specific solution:

$$y(x) = e^x \left[\cos(2x) - \sin(2x)\right].$$

LESSON 17

1. Repeated Real Roots

Derivation 17.1. Consider the second order differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

For simplicity, assume $a = 1$. This is a safe assumption since we can always divide the given equation by a to obtain the equivalent:

$$\frac{d^2y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{a} = 0.$$

If the characteristic equation can be factored as:

$$s^2 + bs + c = (s - r_1)(s - r_2) = 0,$$

then we have:

$$\begin{aligned} b &= -(r_1 + r_2) \\ c &= r_1 r_2. \end{aligned}$$

We assume $r_1, r_2 \in \mathbb{R}$. That means we can rewrite the ODE as:

$$\frac{d^2y}{dx^2} - (r_1 + r_2)\frac{dy}{dx} + r_1 r_2 y = 0.$$

Expanding this we have:

$$\left(\frac{d^2y}{dx^2} - r_2\frac{dy}{dx}\right) - r_1\frac{dy}{dx} + r_1 r_2 y = \left(\frac{d^2y}{dx^2} - r_2\frac{dy}{dx}\right) - r_1\left(\frac{dy}{dx} - r_2 y\right) = 0.$$

Let:

$$v = \frac{dy}{dx} - r_2 y. \tag{17.1}$$

Then:

$$\frac{dv}{dx} = \frac{d^2y}{dx^2} - r_2\frac{dy}{dx}.$$

So we can rewrite the original ODE as:

$$\frac{dv}{dx} - r_1 v = 0.$$

Thus:

$$v(x) = C_1 \exp(r_1 x).$$

Here C_1 is just a constant of integration. Now we can use Eq. (17.1) to see:

$$\frac{dy}{dx} - r_2 y = C_1 \exp(r_1 x).$$

This can be solved with the integrating factor:

$$\mu(x) = \exp(-r_2 x),$$

where we see:

$$y(x) = \frac{1}{\exp(-r_2 x)}\left(\int C_1 \exp(r_1 x)\exp(-r_2 x)\,dx\right) = e^{r_2 x}\left(\int C_1 \exp[(r_1 - r_2)x]\,dx\right).$$

If $r_1 \neq r_2$, we get the result in Theorem 16.11, which you can check. If $r_1 = r_2 = r$ we get:

$$y(x) = e^{rx}\left(\int C_1\,dx\right) = e^{rx}\left(C_1 x + C_2\right) = C_1 x e^{rx} + C_2 e^{rx}.$$

We have proved a theorem.

Theorem 17.2 (Two Repeated Roots). *Consider the second order differential equation:*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

If the characteristic equation $as^2 + bs + c = 0$ has a single root $r \in \mathbb{R}$, then the general solution to the ODE is:

$$y(x) = C_1 x e^{rx} + C_2 e^{rx}.$$

Example 17.3. Consider the ODE:

$$y'' + 2y' + y = 0.$$

The characteristic equation is $s^2 + 2s + 1 = (s+1)^2$. Therefore, $r = 1$ is the only root. The solution to the ODE is therefore:

$$y(x) = C_1 x e^{-x} + C_2 e^{-x}.$$

If we have $y(0) = 1$ and $y'(0) = 2$ then:

$$y(0) = C_2 = 1$$

and

$$y'(x) = C_1 e^{-x} - C_1 x e^{-x} - C_2 e^{-x},$$

so:

$$y'(0) = C_1 - C_2 = C_1 - 1 = 2.$$

Therefore $C_1 = 3$ and the specific solution is:

$$y(x) = 3xe^{-x} + e^{-x}.$$

2. Existence, Uniqueness and the Wronskian

Remark 17.4. Notice in all our solutions to second order linear ODE's, we have found two solutions with two arbitrary constants that have been combined to form a single general solution. The next theorem quantifies when the more general linear second order system has a solution.

THEOREM 17.5. *Consider the linear second order initial value problem:*

$$y'' + p(x)y' + q(x)y = f(x) \qquad y(x_0) = y_0,\, y'(x_0) = y_1.$$

If p, q and g are continuous on the interval $I \subseteq \mathbb{R}$ with $x_0 \in I$, then there exists a unique solution $\varphi(x)$ satisfying the initial value problem. □

Derivation 17.6 (Wronskian)**.** Consider the ODE of form:

$$y'' + p(x)y' + q(x)y = 0$$

This ODE is linear and so we know that if we have two solutions $\varphi_1(x)$ and $\varphi_2(x)$, we can combine them to form an infinite family of solutions:

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

We've already seen this several times. For example in the ODE from Example 16.12:

$$y'' + 3y' + 2y = 0$$

has two solutions $\varphi_1(x) = e^{-x}$ and $\varphi_2(x) = e^{-2x}$, which we combined to form $y(x) = C_1 e^{-x} + C_2 e^{-2x}$. We now derive a condition to ensure the IVP problem:

$$y'' + p(x)y' + q(x)y = 0 \qquad y(x_0) = y_0,\, y'(x_0) = y_1.$$

has a solution of form $C_1\varphi_1(x) + C_2\varphi_2(x)$. That is, we are going to ensure we can solve for explicit values for C_1 and C_2. We require:

$$C_1\varphi_1(x_0) + C_2\varphi_2(x_0) = y_0$$
$$C_1\varphi_1'(x_0) + C_2\varphi_2'(x_0) = y_1$$

If we solve for C_1 and C_2 we see:

$$C_1 = \frac{y_0\varphi_2'(x_0) - y_1\varphi_2(x_0)}{\varphi_1(x_0)\varphi_2'(x_0) - \varphi_2(x_0)\varphi_1'(x_0)}$$
$$C_2 = \frac{y_1\varphi_1(x_0) - y_0\varphi_1'(x_0)}{\varphi_1(x_0)\varphi_2'(x_0) - \varphi_2(x_0)\varphi_1'(x_0)}$$

Notice the term:

$$\varphi_1(x_0)\varphi_2'(x_0) - \varphi_2(x_0)\varphi_1'(x_0)$$

appears in both C_1 and C_2. It is both necessary and sufficient that this term not be zero to ensure the IVP has a solution.

Definition 17.7 (Wronskian)**.** Let $\varphi_1(x)$ and $\varphi_2(x)$ be two functions. The *Wronskian* of the function is:

$$W(\varphi_1, \varphi_2) = \varphi_1\varphi_2' - \varphi_2\varphi_1',$$

which you can also write as:

$$W(\varphi_1(x), \varphi_2(x)) = \varphi_1(x)\varphi_2'(x) - \varphi_2(x)\varphi_1'(x),$$

if you prefer.

Theorem 17.8. *Given two solutions $\varphi_1(x)$ and $\varphi_2(x)$ to the IVP:*

$$y'' + p(x)y' + q(x)y = 0 \qquad y(x_0) = y_0,\ y'(x_0) = y_1.$$

there are constant C_1 and C_2 so that $C_1\varphi_1(x)+C_2\varphi_2(x)$ solves the IVP if and only if $W[\varphi_1, \varphi_2]$ evaluated at x_0 is not equal to zero. □

Example 17.9. Notice in our problem:

$$y'' + 3y' + 2y = 0$$

we have $\varphi_1(x) = e^{-x}$ and $\varphi_2(x) = e^{-2x}$. The Wronskian is:

$$e^{-x}\left(-2e^{-2x}\right) - e^{-2x}\left(-e^{-x}\right) = -2e^{-3x} + e^{-3x} = -e^{-3x}.$$

This function is never zero, so no matter where the initial conditions might occur, the IVP with ODE $y'' + 3y' + 2y = 0$ always has a solution.

Derivation 17.10. Notice:

$$W[\varphi_1, \varphi_2] = 0 \iff \frac{d}{dx}\left[\frac{\varphi_1(x)}{\varphi_2(x)}\right] = \frac{\varphi_1'\varphi_2 - \varphi_2'\varphi_1}{\varphi_2^2} = 0,$$

by the quotient rule. But if:

$$\frac{d}{dx}\left[\frac{\varphi_1(x)}{\varphi_2(x)}\right] = 0,$$

then:

$$\frac{\varphi_1(x)}{\varphi_2(x)} = C,$$

for some constant C. Thus the Wronskian tells us whether two functions are constant multiples of each other.

Example 17.11. If $\varphi_1(x) = e^x$ and $\varphi_2(x) = 2e^x$, then the Wronskian is:

$$e^x\left(2e^x\right) - 2e^x\left(e^x\right) = 0.$$

Definition 17.12 (Linear Independence)**.** A set of functions $\varphi_1, \ldots, \varphi_n$ is linearly independent if for a set of constants $\alpha_1, \ldots, \alpha_n$ we have

$$\alpha_1\varphi_1(x) + \cdots \alpha_n\varphi_n(x) = 0,$$

just in case $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Remark 17.13. In a more general sense, the Wronskian tells us whether two functions are *linearly independent* (Wronskian is not zero) or *linearly dependent* (Wronskian is zero). As we will see, only linearly independent solutions can be combined into a general solution.

LESSON 18

1. More on the Wronskisan

THEOREM 18.1. *Consider the ODE:*

$$y'' + p(x)y' + q(x)y = 0$$

and let $y_1(x)$ *and* $y_2(x)$ *be two solutions. The solution* $C_1y_1(x)+C_2y_2(x)$ *is the* <u>*general solution*</u> *if and only if there is some* x_0 *where* $W[y_1(x_0), y_2(x_0)] \neq 0$*.*

PROOF. The $\varphi(x)$ be a specific solution. Suppose at x_0, $W[y_1(x_0), y_2(x_0)] \neq 0$. Consider the IVP:

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = y_0 = \varphi(x_0),\ y'(x_0) = y_1 = \varphi'(x_0). \tag{18.1}$$

Then by Theorem 17.8, there is C_1 and C_2 such that $C_1y_1(x) + C_2y_2(x)$ solves this IVP. We know that $\varphi(x)$ must also be a solution to this IVP. Applying Theorem 17.5, the solution to this IVP must be unique therefore:

$$\varphi(x) = C_1y_1(x) + C_2y_2(x).$$

Since we choose $\varphi(x)$ to be an arbitrary specific solution, it follows that any solution can be expressed as $C_1y_1(x) + C_2y_2(x)$.

To prove the converse, assume that $W[y_1(x_0), y_2(x_0)] = 0$ for all x_0. Then for any specific solution $\varphi(x)$, we can find no C_1 and C_2 so that $C_1y_1(x) + C_2y_2(x)$ solves Eq. (18.1), yet clearly $\varphi(x)$ solves this problem. Therefore, $\varphi(x)$ does not have form $C_1y_1(x) + C_2y_2(x)$ and this is not a general solution. $\square$

Remark 18.2. For those who have taken matrices (or some linear algebra) this result states that for second order linear ODE's, the solution space is two dimensional made up of a basis of functions y_1 and y_2.

THEOREM 18.3 (Abel's Theorem*). *Consider the following ODE:*

$$y'' + p(x)y' + q(x)y = 0.$$

The Wronskian is given by:

$$W = A\exp\left(-\int p(x)\,dx\right),$$

for some constant A*. In particular, the Wronskian is zero if and only if this constant is zero.*

PROOF. Let $\varphi_1(x)$ and $\varphi_2(x)$ be two solutions. Then:

$$\varphi_1'' + p(x)\varphi_1' + q(x)\varphi_1 = 0$$
$$\varphi_2'' + p(x)\varphi_2' + q(x)\varphi_2 = 0.$$

Multiply the first equation by $-\varphi_2$ and the second by φ_1 to obtain:

$$
\begin{aligned}
&-\varphi_2\varphi_1'' - p(x)\varphi_2\varphi_1' - q(x)\varphi_2\varphi_1 = 0\\
&\varphi_1\varphi_2'' + p(x)\varphi_1\varphi_2' + q(x)\varphi_2\varphi_2 = 0.
\end{aligned}
$$

Add these two equations to see:

$$
\varphi_1\varphi_2'' - \varphi_2\varphi_1'' + (\varphi_1\varphi_2' - \varphi_2\varphi_1')\,p(x) = \varphi_1\varphi_2'' - \varphi_2\varphi_1'' + Wp(x) = 0. \tag{18.2}
$$

Now note:

$$
\frac{d}{dx}W(\varphi_1, \varphi_2) = \frac{d}{dx}\left(\varphi_1\varphi_2' - \varphi_2\varphi_1'\right) = \varphi_1\varphi_2'' - \varphi_2\varphi_1''.
$$

So Eq. (18.2) is:

$$
W' + p(x)W = 0.
$$

This is a separable ODE with a solution:

$$
W = A\exp\left(-\int p(x)\,dx\right).
$$

Moreover, since $\exp[p(x)]$ is always positive, it follows that W is zero if and only if $A = 0$. □

Remark 18.4. In particular, the constant A from Theorem 18.3 depends entirely on what φ_1 and φ_2 are chosen.

Example 18.5. Consider the ODE from Example 17.9:

$$
y'' + 3y' + 2y = 0
$$

where we have $\varphi_1(x) = e^{-x}$ and $\varphi_2(x) = e^{-2x}$. Here $p(x) = 3$, s we expect the Wronskian to be:

$$
W = A\exp\left(-\int 3\,dx\right) = A\exp(-3x).
$$

This agrees with our explicit computation of the Wronskian from Example 17.9, where we had $W = e^{-3x}$. Thus for this choice of functions we can conclude that $A = 1$.

2. Harmonic Oscillators and Conservation of Energy

Remark 18.6. Material in this section can be found in even greater detail in [Mar13].

Derivation 18.7 (Harmonic Oscillator)**.** Consider a mass on a spring resting on a frictionless surface in a vacuum. The only force on the mass is caused by the spring when it is out of

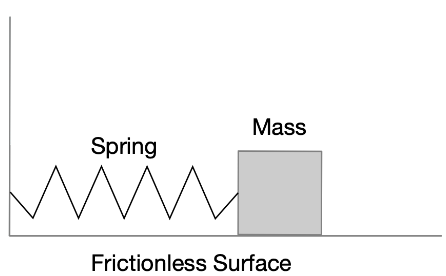


FIGURE 18.1. A mass on a spring on a frictionless surface in a vacuum will experience forces governed only by Hook's law.

equilibrium (stretched or compressed). Recall Hook's law states that the force from a spring is opposite to its displacement from equilibrium. Written as an equation we have:

$$F = -kx,$$

where x is the displacement distance from the spring's equilibrium (which would be $x = 0$).

Ignoring the mass of the spring we can write Newton's law as:

$$m\ddot{x} = -kx,$$

where m is the mass of the block in Fig. 18.1 and x is the position of the block determining how far the spring is stretched or compressed. This ODE is then:

$$\ddot{x} + \frac{k}{m}x = 0.$$

This is called the *Harmonic Oscillator* equation. The characteristic equation for this ODE is:

$$s^2 + \frac{k}{m}s = 0,$$

which has solutions: The characteristic equation for this ODE is:

$$s^2 + \frac{k}{m} = 0,$$

$$s = \pm i\sqrt{\frac{k}{m}}.$$

In this case, we know that the general solution is:

$$x(t) = C_1 \cos\left(t\sqrt{\frac{k}{m}}\right) + C_2 \sin\left(t\sqrt{\frac{k}{m}}\right).$$

If $x(0) = x_0$ and $\dot{x}(0) = v_0$ (the initial velocity), then we can solve:

$$x_0 = x(0) = C_1 \cos(0) = C_1.$$

We also know:

$$v_0 = \dot{x}(0) = \sqrt{\frac{k}{m}} \cdot C_2 \cos(0) \implies C_2 = \frac{v_0}{\sqrt{k/m}}.$$

For simplicity, let $\omega = \sqrt{k/m}$. This is the *angular frequency* of the oscillation. From the angle sum trigonometric identities, we know:

$$A\cos(\omega t - \phi) = A\cos(\phi)\cos(\omega t) + A\sin(\phi)\sin(\omega t).$$

Then we can associate:

$$A\cos(\phi) = x_0$$
$$A\sin(\phi) = \frac{v_0}{\omega}$$

Then:

$$A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2 = x_0^2 + \frac{v_0^2}{\omega^2}.$$

We conclude that:

$$A = \sqrt{x_0^2 + (v_0/\omega)^2} = \sqrt{C_1^2 + C_2^2}.$$

Likewise:

$$\frac{A\sin(\phi)}{A\cos(\phi)} = \tan(\phi) = \frac{v_0}{\omega x_0} = \frac{C_2}{C_1}.$$

The constant ϕ is called the *phase* of the oscillation and we can express the solution of the ODE as:

$$x(t) = A\cos(\omega t - \phi)$$

Notice:

$$\dot{x} = -A\omega\sin(\omega t - \phi).$$

We can compute the energy of the system. The kinetic energy is $K = \frac{1}{2}mv^2$. That is:

$$K = \frac{1}{2}m\left[-A\omega\sin(\omega t - \phi)\right]^2 = \frac{Am\omega^2}{2}\sin^2(\omega t - \phi).$$

Meanwhile by Hook's law (again), the potential energy is $U = \frac{1}{2}kx^2$. That is:

$$U = \frac{1}{2}k\left[A\cos(\omega t - \phi)\right]^2 = \frac{A^2k}{2}\cos^2(\omega t - \phi).$$

Recall: $\omega^2 = k/m$

$$\frac{Am\omega^2}{2} = \frac{Ak^2}{2}.$$

At last we can compute:

$$K + U = \frac{Am\omega^2}{2}\sin^2(\omega t - \phi) + \frac{A^2k}{2}\cos^2(\omega t - \phi) = \\ \frac{A^2k}{2}\sin^2(\omega t - \phi) + \frac{A^2k}{2}\cos^2(\omega t - \phi) = \frac{A^2k}{2}.$$

This is a constant and shows that in the Harmonic Oscillator, total energy is conserved.

Remark 18.8. This is not an accident. We show this is a result of a more general property of second order ODE's.

THEOREM 18.9 (Energy Theorem). *Consider the second order (autonomous) ODE:*

$$\ddot{x} = f(x). \tag{18.3}$$

Let:

$$F(x) = \int f(x)\,dx$$

be the anti-derivative of $f(x)$ with no constant of integration and let:

$$\mathcal{E}(t) = \frac{1}{2}\dot{x}^2(t) - F[x(t)]$$

be the energy of the system modeled by the ODE. Then $\mathcal{E}(t)$ is constant. That is ODE's with form Eq. (18.3) exhibit conservation of energy.

PROOF. To show $\mathcal{E}(t)$ is constant, we take it's derivative:

$$\frac{d\mathcal{E}}{dt} = \dot{x}\ddot{x} - F'[x(t)]\dot{x} = \dot{x}\left[\ddot{x} - f(x)\right] = 0,$$

since $\ddot{x} - f(x) = 0$ by Eq. (18.3). Therefore $\mathcal{E}'(t) = 0$ and $\mathcal{E}(t)$ must be a constant. $\square$

Remark 18.10. In the harmonic oscillator we had:

$$\ddot{x} = f(x) = -\frac{k}{m}x.$$

Thus: $F(x) = -\frac{1}{2m}kx^2$ and so:

$$\mathcal{E} = \frac{1}{2}v^2 - \left(-\tfrac{1}{2m}kx^2\right) = \frac{1}{2}v^2 + \frac{1}{2m}kx^2,$$

where $v = \dot{x}$. This is constant by the theorem and so is the mass m. Then:

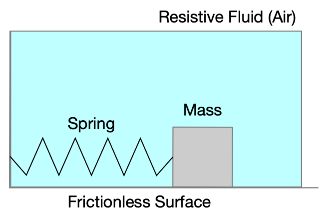
$$E = m\mathcal{E} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

is also a constant and we have recovered the ordinary kinetic and potential energy of the mass-spring system along with the principle of energy conservation from mechanics.

LESSON 19

1. Damped Oscillation

Example 19.1 (Damped Oscillator)**.** Starting from our mass on a spring, assume the mass is surrounded by a fluid (air) that resists its motion. In this case, there is still no surface



friction. The fluid (air) resistance we studied in Example 4.9 was more appropriate for fast moving bodies. For slow moving bodies (like our mass on a spring) we will use the stokes force, which is proportional to $\dot{y}$ (as opposed to $\dot{y}^2$). Summing up the forces we have:

$$m\ddot{y} = -b\dot{y} - ky,$$

or:

$$m\ddot{y} + b\dot{y} + ky = 0.$$

This second order ODE has characteristic polynomial:

$$ms^2 + bs + k = 0,$$

with solutions:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are now three possibilities for the discriminant $\Delta = b^2 - 4mk$.

Overdamped Motion: When $b^2 - 4mk > 0$, this implies r_1 and r_2 are real. Moreover, both roots must be negative since $-b > \sqrt{b^2 - 4mk}$, since $m, k > 0$. The solution is:

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

From this, we conclude that $\lim_{t\to\infty} y(t) = 0$. That is, the damping effect of the fluid stops the motion in the long run. Taking the derivative we have:

$$\dot{y} = r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t}$$

If we factor this expression, we have:

$$\dot{y} = e^{r_1 t}\left(r_1 C_1 + r_2 C_2 e^{(r_1 - r_2)t}\right)$$

If we set this equal to zero (to find a local maximum) we see that if the derivative is zero then:

$$C_1 r_1 + r_2 C_2 e^{(r_1 - r_2)t} = 0,$$

which has only one solution:

$$t^* = \frac{1}{r_1 - r_2} \log\left(-\frac{r_1 C_1}{r_2 C_2}\right).$$

We conclude that if $t^* > 0$, there must be one maximum or one minimum and no oscillations. This is called *overdamped* motion. An example of overdamped motion is shown in Fig. 19.1 with the equation $\ddot{y} + 4y' + y = 0$.

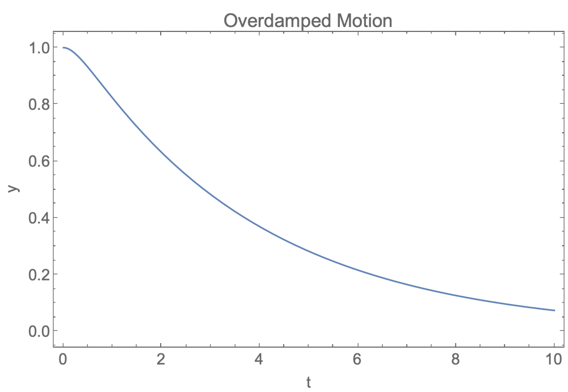


FIGURE 19.1. The overdamped oscillator slowly returns to the equilibrium position $y = 0$.

Underdamped Motion: When $b^2 - 4mk < 0$, this implies r_1 and r_2 are complex. We can write:

$$r_1, r_2 = \frac{-b}{2m} \pm i\frac{\sqrt{4mk - b^2}}{2m}.$$

Let:

$$\alpha = \frac{b}{2m}$$
$$\omega = \frac{\sqrt{4mk - b^2}}{2m}.$$

We know $\alpha > 0$ and the roots are $r_1, r_2 = -\alpha \pm i\omega$ and we know the solution is:

$$y(t) = e^{-\alpha t}\left[C_1 \cos(\omega t) + C_2 \sin(\omega t)\right].$$

Just as in the case of the harmonic oscillator we can write this as:

$$y(t) = e^{-\alpha t}\left[A \cos(\omega t - \phi)\right],$$

with $A = \sqrt{C_1^2 + C_2^2}$ and $\tan(\phi) = C_2/C_1$. We see that $\lim_{t\to\infty} = 0$ because $\lim_{t\to\infty} e^{-\alpha t} = 0$ and the cosine term is bounded between -1 and 1. Thus, process exhibits exponentially damped oscillation. This motion is called *underdamped* because there is not enough damping to present oscillation unlike in the overdamped case. An example of underdamped motion is shown in Fig. 19.2 with the equation $\ddot{y} + \frac{1}{2}y' + y = 0$. We also show the exponential envelope controlling the decay. These are computed as $y_{\text{env}}(t) = Ae^{-\alpha t}$.

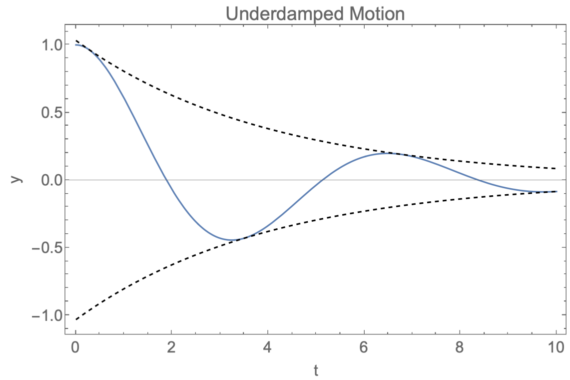


FIGURE 19.2. The underdamped oscillator converges to the equilibrium point $y = 0$ while continuing to oscillate.

Critically Damped Motion: When $b^2 - 4mk = 0$, this implies $r_1 = r_2 = r = -b/(2m)$ and so there is one real negative root and the solution is:

$$y(t) = C_1 e^{-\frac{b}{2m}t} + C_2 t e^{-\frac{b}{2m}t} = (C_1 + C_2 t)e^{-\frac{b}{2m}t}.$$

Then:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{C_1 + C_2 t}{\exp\left(\frac{b}{2m}t\right)} = 0,$$

since exponential functions dominate linear functions. (Alternatively, you can use L'Hospital's Rule on this limit.) Likewise:

$$\dot{y} = \left[C_2 - \frac{b}{2m}(C_1 + C_2 t)\right] \exp\left(-\tfrac{b}{2m}t\right).$$

As before, this function has one root at:

$$t^* = \frac{bC_1 - 2mC_2}{bC_2}.$$

Therefore there is one maximum or minimum at most and no oscillations. An example of critically damped motion is shown in Fig. 19.3 with the equation $\ddot{y} + 2y' + y = 0$. Notice the convergence to the equilibrium $y(t) = 0$ is faster than in the overdamped case because the fluid is less resistive to the motion of the mass.

2. Circuits - Part 3

Example 19.2 (RLC Circuits*)**.** Consider a circuit consisting of a battery, a capacitor, an inductor and a resistor in series. Recall from Ohm's law (Eq. (15.2)) we have:

$$V_R = RI.$$

The equation for capacitance (Eq. (15.1)) tell us:

$$V_C = \frac{Q}{C}.$$

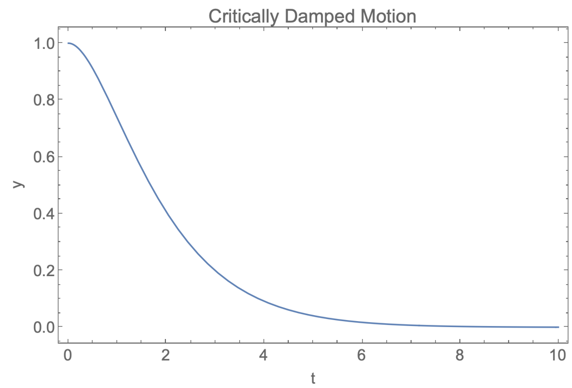


FIGURE 19.3. The critically damped oscillator returns to the equilibrium without oscillation and does so more quickly than in the overdamped case.

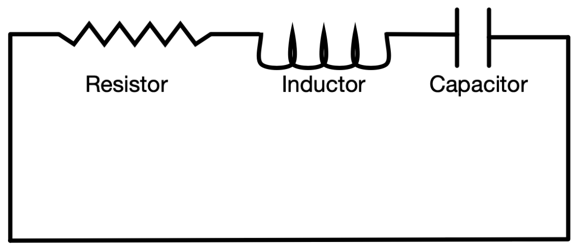


FIGURE 19.4. An RLC circuit consists of a capacitor, inductor and resistor in series with a battery.

Lastly, we recall from the inductance law that:

$$V_L = V_0 - L\dot{I}.$$

We know that the total voltage in the circuit must be $V = V_L + V_R + V_C$. Combining the voltages we know:

$$V_0 = L\dot{I} + RI + \frac{Q}{C}$$

Then differentiating this expression with respect to t (which removes Q) yields:

$$0 = L\ddot{I} + R\dot{I} + \frac{1}{C}\frac{dQ}{dt} = L\ddot{I} + RI + \frac{1}{C}I,$$

or

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}\frac{dQ}{dt} = 0.$$

When:

$$\Delta = \left(\frac{R}{L}\right)^2 - \frac{4}{LC} < 0,$$

the current oscillates (in direction) in the circuit and this is the electrical analog of a mass on a spring in a viscous fluid with the resistor playing the role of the fluid, the capacitor playing the role of the mass and the inductor playing the role of the spring. The oscillating current is called *alternating current*. More details on this example can be found in [**PM13**].

LESSON 20

1. Method of Undetermined Coefficients - Polynomial Forcing Functions

Lemma 20.1. *Suppose that $\varphi_1(x)$ solves $y''+ay'+by=q_1(x)$ and $\varphi_2(x)$ solves $y''+ay'+by=q_2(x)$. Then $K_1\varphi_1(x)+K_2\varphi_2(x)$ solves $y''+ay'+by=K_1q_1(x)+K_2q_2(x)$.*

PROOF. Let $\varphi(x)=K_1\varphi_1(x)+K_2\varphi_2(x)$. Direct computation shows:

$$\frac{d^2\varphi}{dx^2}+a\frac{d\varphi}{dx}+b\varphi = K_1\left(\varphi_1''+a\varphi_1'+b\varphi_1\right)+K_2\left(\varphi_2''+a\varphi_2'+b\varphi_2\right)=K_1q_1(x)+K_2q_2(x)$$

□

Lemma 20.2. *Suppose $\varphi(x)$ solves $y''+ay'+by=0$ and $\psi(x)$ solves $y''+ay'+by=q(x)$. Then $\varphi(x)+\psi(x)$ also solves $y''+ay'+by=q(x)$.*

PROOF. We can verify this by noting:

$$\frac{d^2\varphi}{dx^2}=-a\frac{d\varphi}{dx}-b\varphi$$
$$\frac{d^2\psi}{dx^2}=-a\frac{d\psi}{dx}-b\psi+q(x).$$

Adding the two equations and factoring yields:

$$\frac{d^2}{dx^2}\left(\varphi+\psi\right)=-a\frac{d}{dx}\left(\varphi+\psi\right)-b\left(\varphi+\psi\right)+q(x).$$

Therefore:

$$\frac{d^2}{dx^2}\left(\varphi+\psi\right)+a\frac{d}{dx}\left(\varphi+\psi\right)+b\left(\varphi+\psi\right)=q(x).$$

□

Remark 20.3. The previous two lemmas tell us two things:

(1) If we have a non-homogeneous equation of the form:

$y''+ay'+by=q_1(x)+q_2(x)$,

we can solve two simpler non-homogeneous equations:

$y''+ay'+by=q_1(x)$

$y''+ay'+by=q_2(x)$

and then add the solutions together.

(2) General solutions of a non-homogeneous equation $y'' + ay' + by = q(x)$ will have form $y(x) = \varphi(x) + \psi(x)$, where $\varphi(x)$ is the *general solution to the homogeneous equation* and $\psi(x)$ is a *particular* to the non-homogeneous equation.

Remark 20.4. General theorems on the *method of undetermined coefficients* will be stated but not proved. The proofs – frankly – are boring. They can be found in [**BC98, NSS11, BDM21, AD12**]. The proofs are just generalizations of the examples we will use to build up the technique.

Example 20.5. Consider the ODE:

$$y'' + 3y' + 2y = x^2.$$

We don't have an idea of how to solve a problem like this, but we can derive a method. Let

$$L = \frac{d^2}{dx^2} + 3\frac{d}{dx} + 2.$$

Our problem is just:

$$L(y) = x^2.$$

Notice the following fact:

$$L\left(ax^2 + bx + c\right) = 2ax^2 + (6a + 2b)x + (2a + 3b + 2c).$$

Notice we applied L to a degree two polynomial and the result was a degree two polynomial. This is useful since we can attempt to find values for a, b and c so that:

$$2ax^2 + (6a + 2b)x + (2a + 3b + 2c) = x^2,$$

This means in particular:

$$2a = 1$$
$$6a + 2b = 0$$
$$2a + 3b + 2c = 0.$$

Notice this system of equations has a triangular structure, which implies it must have a solution. To see this note, the value of a is determined, the value of b is determined from the value of a and the value of c is determined from the values of a and b. We conclude that:

$$a = \tfrac{1}{2}$$
$$b = \tfrac{-3}{2}$$
$$c = \tfrac{7}{4}$$

Thus, the specific solution to this problem is:

$$\psi(x) = \tfrac{1}{2}x^2 - \tfrac{3}{2}x + \tfrac{7}{4}.$$

For the homogeneous equation $L(y) = 0$, we know the characteristic equation is:

$$s^2 + 3s + 2 = (s + 1)(s + 2) = 0.$$

This implies the general solution to the homogeneous equation is:

$$\varphi(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

We deduce the solution to the problem is then:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \tfrac{1}{2}x^2 - \tfrac{3}{2}x + \tfrac{7}{4}.$$

We can generalize two results from this example.

Lemma 20.6. *Define the linear operator:*

$$L = \frac{d^2}{dx^2} + \alpha \frac{d}{dx} + \beta.$$

If $P(x)$ is a degree n polynomial with

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

then $L[P(x)]$ is a degree n polynomial with:

$$L[P(x)] = \beta a_n x^n + (\beta a_{n-1} + \alpha n a_n) x^{n-1} + \cdots .$$

Thus for any constraint of the form

$$L[P(x)] = x^n,$$

there is a unique solution with $a_n = \frac{1}{\beta}$ and a_k determined solely by the values $a_{k+1}, \dots, a_n$. □

Theorem 20.7. *The non-homogeneous equation:*

$$y'' + ay' + by = x^n$$

has specific solution with form:

$$\psi(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

where the coefficients can be determined from the result in Lemma 20.6. □

Remark 20.8. This method of solving a non-homogeneous equation is called *the method of undetermined coefficients*. It can be generalized further.

2. Method of Undetermined Coefficients - Polynomial-Exponential Forcing Functions - Part 1

Example 20.9. Consider the ODE:

$$y'' + 3y' + 2y = x^2 e^x.$$

Again let:

$$L = \frac{d^2}{dx^2} + 3\frac{d}{dx} + 2.$$

Notice that:

$$L\left[\left(ax^2 + bx + c\right) e^x\right] = \left[6ax^2 + (10a + 6b)x + (2a + 5b + 6c)\right] e^x.$$

Here the multiple e^x comes from the fact that it appears on the right-hand-side of the ODE. Just as before we applied L to a degree two polynomial multiplied by e^x and the result was a degree two polynomial multiplied by e^x. As before, we can equate the coefficients on the desired right-hand-side $x^2 e^x$ with the coefficients on the output to see we need:

$$\begin{aligned} 6a &= 1 \\ 10a + 6b &= 0 \\ 2a + 5b + 6c &= 0. \end{aligned}$$

Again this system of equations is triangular and so we know there must be a unique solution.

$$a = \frac{1}{6}$$
$$b = \frac{-5}{18}$$
$$c = \frac{19}{108}$$

The solution to the ODE is then:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{6}x^2 - \frac{5}{18}x + \frac{19}{108}.$$

Remark 20.10. We see in the next example that we cannot immediately generalize this result without observing a second possibility.

LESSON 21

1. Method of Undetermined Coefficients - Polynomial-Exponential Forcing Functions - Part 2

Example 21.1. Consider the ODE:

$$y'' + 3y' + 2y = x^2 e^{-x}.$$

Again let:

$$L = \frac{d^2}{dx^2} + 3\frac{d}{dx} + 2.$$

In the last example we assumed that:

$$\psi(x) = \big(ax^2 + bx + c\big)\, e^{-x}$$

Again, the multiple e^{-x} comes from the right-hand-side. This time however, when we apply the linear operator we see:

$$L\left[\big(ax^2 + bx + c\big)\, e^{-x}\right] = (2a + b + 2ax)e^{-x}.$$

Somehow, we have lost all the x^2 terms, which we need in order to find a specific solution to the ODE because the right-hand-side of the ODE is $x^2 e^{-x}$.

Lemma 21.2. *Suppose:*

$$L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b.$$

and let $h(x)$ be a function of x. Then:

$$L\left[e^{\alpha x} h(x)\right] = L(e^{\alpha x})h(x) + (2\alpha + a)\, e^{\alpha x} h'(x) + e^{\alpha x} h''(x) = e^{\alpha x} \tilde{L}[h(x)],$$

where:

$$\tilde{L} = \frac{d^2}{dx^2} + (2\alpha + a)\frac{d}{dx} + (\alpha^2 + a\alpha + b). \tag{21.1}$$

is a second operator.

Remark 21.3. Before we do the proof, notice that if $h(x)$ is a polynomial of degree n and $e^{\alpha x}$ is a solution to the homogeneous equation $L(y) = 0$, then we see the term:

$$L(e^{\alpha x})h(x) = 0.$$

In particular, this means that the polynomial $\tilde{L}[h(x)]$ does not have degree n. This is what we just observed in Example 21.1.

Proof of Lemma 21.2. Applying L to $e^{\alpha x}h(x)$ we have:

$$\frac{d^2}{dx^2}\left[e^{\alpha x}h(x)\right] = \alpha^2 e^{\alpha x}h(x) + 2\alpha e^{\alpha x}h'(x) + e^{\alpha x}h''(x)$$
$$a\frac{d}{dx}\left[e^{\alpha x}h(x)\right] = a\alpha e^{\alpha x}h(x) + ae^{\alpha x}h'(x)$$

Then:

$$L\left[e^{\alpha x}h(x)\right] = \left(\alpha^2 e^{\alpha x} + a\alpha e^{\alpha x} + be^{\alpha x}\right)h(x) + \left(2\alpha e^{\alpha x} + ae^{\alpha x}\right)h'(x) + e^{\alpha x}h''(x). \tag{21.2}$$

Computation shows that:

$$L\left(e^{\alpha x}\right) = \alpha^2 e^{\alpha x} + a\alpha e^{\alpha x} + be^{\alpha x}.$$

Therefore:

$$L\left[e^{\alpha x}h(x)\right] = L(e^{\alpha x})h(x) + (2\alpha + a)\,e^{\alpha x}h'(x) + e^{\alpha x}h''(x).$$

Factoring Eq. (21.2) differently yields:

$$L\left[e^{\alpha x}h(x)\right] = \left[h''(x) + (2\alpha + a)h'(x) + \left(\alpha^2 + a\alpha + b\right)\right]e^{\alpha x},$$

which can be written as:

$$L\left[e^{\alpha x}h(x)\right] = e^{\alpha x}\tilde{L}[h(x)],$$

where $\tilde{L}$ is defined in Eq. (21.1). □

Corollary 21.4. *Suppose that $e^{\alpha x}$ solves $L(y) = 0$ and α is a repeated root. Then*

$$L\left[e^{\alpha x}h(x)\right] = h''(x)e^{\alpha x}.$$

Proof. Recall from Derivation 17.1 that we can write the operator L as

$$\frac{d^2}{dx^2} - (r_1 + r_2)\frac{d}{dx} + r_1 r_2 y = 0,$$

where r_1 and r_2 are the roots of the characteristic polynomial of the operator L. We assume that $r_1 = r_2 = r$ and using the definition of the operator L from the theorem:

$$L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b,$$

we see:

$$a = -(r_1 + r_2) = -2r = -2\alpha.$$

Because we assume α is the root of the characteristic polynomial. Therefore, from the theorem we have:

$$L\left[e^{\alpha x}h(x)\right] = \left[h''(x) + (2\alpha - 2\alpha)h'(x) + \left(\alpha^2 + a\alpha + b\right)\right]e^{\alpha x}.$$

The characteristic polynomial is: $s^2 + as + b$ and therefore $\alpha^2 + a\alpha + b = 0$ and we conclude:

$$L\left[e^{\alpha x}h(x)\right] = h''(x)e^{\alpha x}.$$

□

Example 21.5. We now reconsider the ODE:

$$y'' + 3y' + 2y = x^2 e^{-x}.$$

Again:

$$L = \frac{d^2}{dx} + 3\frac{d}{dx} + 2.$$

We can see that e^{-x} is a solution to $y'' + 3y' + 2y = 0$. If $h(x)$ is a degree n polynomial, we know that: $L[h(x)e^{-x}] = k(x)e^{-x}$, where $k(x)$ is a degree $n-1$ polynomial. Therefore, we need $h(x)$ to be a degree 3 polynomial in order to ensure $k(x)$ has degree 2 and we can apply the approach method of undetermined coefficients. Therefore consider:

$$L\left[\left(ax^3 + bx^2 + cx + d\right)e^{-x}\right] = \left[3ax^2 + (6a + 2b)x + (2b + c)\right]e^{-x}.$$

First, notice that d has vanished on the right-hand-side. Therefore we are free to set $d = 0$. We can tell this will always be the case using Lemma 21.2 in which only $h'(x)$ is retained.

We can now construct the equations:

$$\begin{aligned} 3a &= 1 \\ 6a + 2b &= 0 \\ 2b + c &= 0 \end{aligned}$$

If we solve these equation, we can now derive the general solution to the ODE as:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \left(\tfrac{1}{3}x^3 - x^2 + 2x\right)e^{-x} = \left(\tfrac{1}{3}x^2 - x + 2\right)xe^{-x}.$$

Example 21.6. Consider now the ODE:

$$y'' + 2y' + y = x^2 e^{-x}.$$

We can see that e^{-x} is a solution to $y'' + 2y' + y = 0$ and moreover, -1 is a repeated root of the characteristic equation $s^2 + 2s + 1 = 0$. If $h(x)$ is a degree n polynomial, we know that: $L[h(x)e^{-x}] = k(x)e^{-x}$, where $k(x)$ is a degree $n-2$ polynomial. Therefore, we need $h(x)$ to be a degree 4 polynomial in order to ensure $k(x)$ has degree 2 and we can apply the approach method of undetermined coefficients. Therefore consider:

$$L\left[\left(ax^4 + bx^3 + cx^2 + dx + f\right)e^{-x}\right] = \left(12ax^2 + 6bx + 2c\right)e^{-x}.$$

Notice that dx and f have vanished on the right-hand-side. Therefore we are free to set $d = 0$ and $f = 0$. We can tell this will always be the case using Corollary 21.4 in which only $h''(x)$ is retained.

We can now construct the equations:

$$\begin{aligned} 12a &= 1 \\ 6b &= 0 \\ 2c &= 0 \end{aligned}$$

If we solve these equation, we can now derive the general solution to the ODE as:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \tfrac{1}{12}x^4 e^{-x}.$$

Remark 21.7. Using these examples and lemmas, we can state a theorem.

Theorem 21.8. *Consider the ODE:*

$$y'' + ay' + by = e^{\alpha x} x^n.$$

The specific solution has form:

$$\psi(x) = x^s \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0\right) e^{\alpha x},$$

where s is the number of times α appears as a root in the characteristic polynomial of the corresponding homogeneous ODE. $\square$

LESSON 22

1. Method of Undetermined Coefficients - Polynomial-Exponential-Sinusoidal Forcing Functions

Example 22.1. Consider the ODE:

$$y'' + 3y' + 2y = \sin(2x).$$

Again:

$$L = \frac{d^2}{dx^2} + 3\frac{d}{dx} + 2.$$

Notice that if we apply L to $\sin(2x)$ we will obtain terms with $\cos(2x)$. Therefore we investigate:

$$L\left[a\cos(2x) + b\sin(2x)\right] = (6b - 2a)\cos(2x) + (-6a - 2b)\sin(2x)$$

From this we derive the two equation:

$$\begin{aligned} 6b - 2a &= 0 \\ -2b - 6a &= 1 \end{aligned}$$

Solving we have $a = -\frac{3}{20}$ and $b = -\frac{1}{20}$. Using this we can deduce that:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} - \frac{3}{20}\cos(2x) - \frac{1}{20}\sin(2x)$$

Example 22.2. Consider the ODE:

$$y'' + 2y' + 2y = e^{-x}\cos(x).$$

Here:

$$L = \frac{d^2}{dx^2} + 2\frac{d}{dx} + 2.$$

We note first that the roots of the characteristic polynomial $s^2 + 2s + 2 = 0$ are: $s = -1 \pm i$. This implies that:

$$\left[C_1 e^{-x}\cos(x) + C_2 e^{-x}\sin(x)\right] e^{-x}$$

is a solution to the homogeneous $y'' + 2y' + 2y = 0$. This means that if we try the solution: $\psi(x) = \left[A\cos(x) + B\sin(x)\right] e^{-x}$ we see $L[\psi(x)] = 0$, which is not very useful. We might have expected this. Notice the ODE can be written as:

$$y'' + 2y' + 2y = \mathrm{Re}\left[e^{(-1+i)x}\right],$$

where $\mathrm{Re}(\cdot)$ denotes the real part of an imaginary number. Notice a root of the characteristic polynomial appears in the exponent. This caused trouble in Examples 21.1 and 21.6.

Following Theorem 21.8, we multiply by x (because only one root appears in the exponential function on the right-hand-side) and see that:

$$L\left\{[ax\cos(x) + bx\sin(x)]\,e^{-x}\right\} = 2be^{-x}\cos[x] - 2ae^{-x}\sin[x]$$

Thus, setting $a = 0$ and $b = \frac{1}{2}$ we have the specific solution:

$$\psi(x) = \frac{1}{2}xe^{-x}\sin(x).$$

We can then combine this with the general solution to the homogeneous equation to see:

$$y(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x) + \frac{1}{2}xe^{-x}\sin(x).$$

Remark 22.3. Just as before, we can generalize these results into a full theorem.

THEOREM 22.4. *Consider the ODE with form:*

$$y'' + ay' + by = e^{\alpha x}x^n\sin(\beta x) \quad \text{or} \quad y'' + ay' + by = e^{\alpha x}x^n\cos(\beta x)$$

The specific solution has form:

$$\psi(x) = x^s\left[\left(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0\right)e^{\alpha x}\cos(\beta x) + \left(b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0\right)e^{\alpha x}\sin(\beta x)\right],$$

where s is the number of times $\alpha + \beta i$ appears as a root in the characteristic polynomial of the ODE. □

Remark 22.5. The proof of the full theorem is very fiddly and a bit dull. In a sense it provides no new insights and is entirely about bookkeeping. Those interested in it can refer to a book on ordinary differential equations.

Remark 22.6. There is a point where finding the solution using the method of undetermined coefficients becomes algebraically unpleasant. It is generally useful only to know the structure of the solution rather than the exact solution. A computer algebra system can be used to find specific solutions, if needed.

2. Forced Oscillation - Undamped Case

Derivation 22.7. Consider the undamped harmonic oscillator but with a forcing function:

$$m\ddot{x} + kx = A\cos(\omega t).$$

Here A is some parameter and ω is the forcing frequency. This time we define:

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

The ODE can be rewritten as:

$$\ddot{x} + \omega_0^2x = A\cos(\omega t),$$

where we are ignoring the fact we have divided by m because A is just a parameter that is provided from outside. The homogeneous equation is

$$\ddot{x} + \omega_0^2x = 0,$$

with operator:

$$L = \frac{d^2}{dt^2} + \omega_0^2 = 0,$$

with characteristic equation $s^2 + \omega_0^2 = 0$. This implies the roots of the characteristic equation are $\pm i\omega_0$. Recall from Derivation 18.7 that the solution to the homogeneous harmonic oscillator is:

$$x(t) = A_0 \cos(\omega_0 t - \phi).$$

First assume that $\omega_0 \neq \omega$. Applying the method of undetermined coefficients to this problem. Using Theorem 22.4 we assume:

$$\psi(t) = a\cos(\omega t) + b\sin(\omega t).$$

Applying L we see:

$$L\left[a\cos(\omega t) + b\sin(\omega t)\right] = a\left(\omega_0^2 - \omega^2\right)\cos(\omega t) + b\left(\omega_0^2 - \omega^2\right)\sin(\omega t).$$

From this, we see that setting $b = 0$ and:

$$a = \frac{A}{\omega_0^2 - \omega^2},$$

will yield the appropriate right-hand-side. We conclude that the specific solution is:

$$\psi(t) = \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t).$$

The general solution is then:

$$x(t) = A_0\cos(\omega_0 t - \phi) + \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t),$$

where A_0 and ϕ are set using initial conditions.

Example 22.8. If we have the specific ODE:

$$\ddot{x} + x = \cos(2t),$$

the solution is:

$$x(t) = A_0\cos(t - \phi) - \frac{1}{3}\cos(2t)$$

Suppose $x(0) = 0$ and $x'(0) = 0$, we need to determine A_0 and ϕ. Plugging in $t = 0$ to $x(t)$ and $x'(t)$ we see:

$$x(0) = A_0\cos(-\phi) - \frac{1}{3} = 0 \implies A_0\cos(-\phi) = \frac{1}{3}$$
$$x'(0) = -A_0\sin(-\phi) = 0 \implies A_0\sin(-\phi) = 0.$$

Then dividing these two terms we see:

$$\tan(-\phi) = 0 \implies \phi = 0.$$

when we assume that $\phi \in [-\pi/2, \pi/2]$ (which is sensible because there would really be an infinite number of equivalent solutions otherwise). Now this implies:

$$A_0\cos(-\phi) = \frac{1}{3} \implies A_0 = \frac{1}{3}.$$

So the solution is:

$$x(t) = \frac{1}{3}\left[\cos(t) - \cos(2t)\right]$$

The solution is shown in Fig. 22.1.

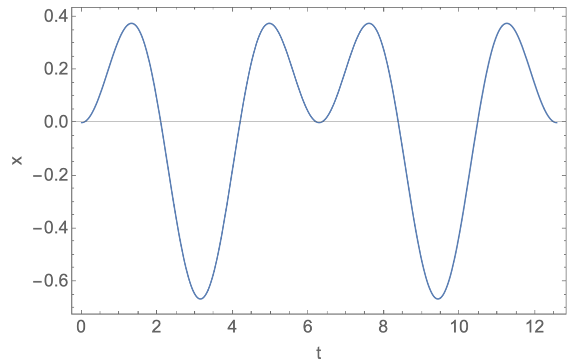


FIGURE 22.1. The forced harmonic oscillator with $\omega_0 = 1$ and $\omega = 2$.

Example 22.9 (Beat Frequency)**.** Suppose that $x(0) = 0$ and $\dot{x}(0) = 0$. That is, the initial velocity of the driven harmonic oscillator is 0 and the oscillator starts at $x_0 = 0$. For these initial conditions, we can deduce that:

$$A_0 = -\frac{A}{\omega_0^2 - \omega^2}$$
$$\phi = 0.$$

The solution is then:

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right].$$

Using a trigonometric identity, we can write:

$$\cos(\omega t) - \cos(\omega_0 t) = 2\sin\left(\frac{\omega_0 - \omega}{2}t\right)\sin\left(\frac{\omega_0 + \omega}{2}t\right).$$

This implies the solution is:

$$x(t) = \frac{2A}{\omega_0^2 - \omega^2}\sin\left(\frac{\omega_0 - \omega}{2}t\right)\sin\left(\frac{\omega_0 + \omega}{2}t\right).$$

When $|\omega_0 - \omega|$ is small, the result is a time varying amplitude:

$$A(t) = \frac{2A}{\omega_0^2 - \omega^2}\sin\left(\frac{\omega_0 - \omega}{2}t\right)$$

that creates an envelope around the primary sinusoid; that is:

$$x(t) = A(t)\sin\left(\frac{\omega_0 + \omega}{2}t\right).$$

The result can be heard as an oscillation in volume and is called a *beat* with *beat frequency* $|\omega_0 - \omega|/2$. This is shown in Fig. 22.2.

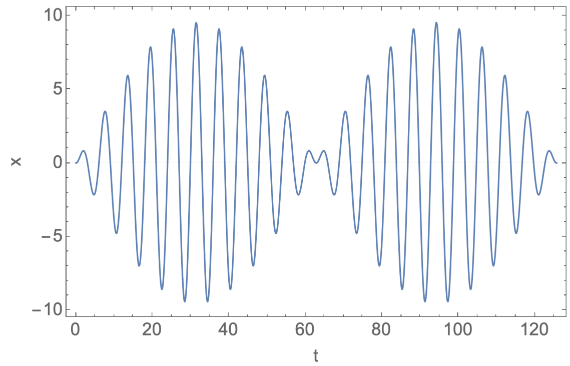


FIGURE 22.2. The forced harmonic oscillator with $\omega_0 = 1$ and $\omega = 10/11$. The resulting sinusoidal envelope that changes the amplitude results in a beat frequency.

LESSON 23

1. Forced Oscillation - Undamped Resonance

Derivation 23.1. Notice in our analysis of the forced undamped oscillator:

$$\ddot{x} + \omega_0^2 x = A\cos(\omega t),$$

we have the expression $\omega_0^2 - \omega^2$ in the denominator of the solution. If $\omega = \omega_0$, we must use a different analysis. Applying Theorem 22.4, the solution must have form:

$$\psi(t) = t\left[a\cos(\omega_0 t) + b\sin(\omega_0 t)\right].$$

Applying the operator:

$$L = \left(\frac{d^2}{dt^2} + \omega_0^2\right)$$

yields:

$$L(\psi) = 2\omega_0\left(b\cos\left(\omega_0 t\right) - a\sin\left(\omega_0 t\right)\right)$$

Thus we set $a = 0$ and:

$$b = \frac{A}{2\omega_0}.$$

The general solution is then:

$$x(t) = A_0\cos(\omega_0 t - \phi) + \frac{A}{2\omega_0} t\sin(\omega_0 t)$$

Notice the amplitude of $x(t)$ grows linearly in time as shown in Fig. 23.1. The linear growth

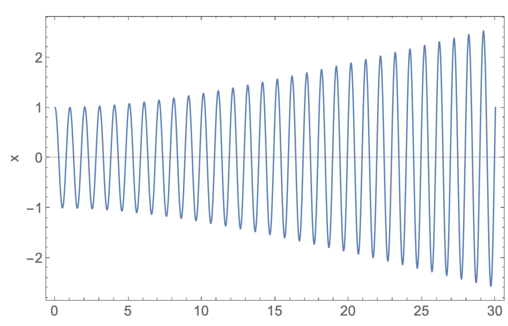


FIGURE 23.1. Resonance in which the amplitude of the solution $x(t)$ grows linearly in time is caused by the driving frequency matching the natural frequency of the harmonic oscillator. Here $\omega = \omega_0 = 2\pi$, $A_0 = 1$, $\phi = 0$ and $A = 2$.

in the amplitude is caused by the fact that the driving frequency matching the natural frequency of the harmonic oscillator. This is called *resonance*.

Remark 23.2. There are a number of phenomena attributed to resonance. However, one we are all familiar with is sloshing tea or coffee while walking. It turns out water in a cup has a resonant frequency that is roughly equal to the frequency of the human gait.

2. Underdamped Oscillation with Forcing

Derivation 23.3. Consider the damped oscillator with forcing:

$$m\ddot{y} + b\dot{y} + ky = A\cos(\omega t).$$

If we assume the un-forced system is underdamped so that $b^2 - 4mk < 0$ the solution to the homogeneous ODE is:

$$\varphi(t) = e^{-\alpha t}\left[A\cos(\omega_0 t - \phi)\right],$$

with:

$$\begin{aligned}\alpha &= \frac{b}{2m}\\ \omega_0 &= \frac{\sqrt{4mk - b^2}}{2m}.\end{aligned}$$

The operator in this case is:

$$L = m\frac{d^2}{dx^2} + b\frac{d}{dx} + k.$$

Let:

$$\psi(t) = r_1\cos(\omega t) + r_2\sin(\omega t).$$

Applying L we have:

$$L[\psi] = \cos(\omega t)\left(br_2\omega + kr_1 - mr_1\omega^2\right) - \left(br_1\omega - kr_2 + mr_2\omega^2\right)\sin(\omega t).$$

We need:

$$\begin{aligned}A &= \left(br_2\omega + kr_1 - mr_1\omega^2\right)\\ 0 &= \left(br_1\omega - kr_2 + mr_2\omega^2\right)\end{aligned}$$

Solving for r_1 and r_2 yields:

$$\begin{aligned}r_1 &= \frac{A\left(k - m\omega^2\right)}{b^2\omega^2 + \left(k - m\omega^2\right)^2}\\ r_2 &= \frac{Ab\omega}{b^2\omega^2 + \left(k - m\omega^2\right)^2}.\end{aligned}$$

We conclude that:

$$\psi(t) = \frac{A\left(k - m\omega^2\right)}{b^2\omega^2 + \left(k - m\omega^2\right)^2}\cos(\omega t) + \frac{Ab\omega}{b^2\omega^2 + \left(k - m\omega^2\right)^2}\sin(\omega t).$$

Thus the solution to the problem is:

$$y(t) = \varphi(t) + \psi(t) = e^{-\alpha t}\left[A\cos(\omega_0 t - \phi)\right] + \frac{A\left(k - m\omega^2\right)}{b^2\omega^2 + \left(k - m\omega^2\right)^2}\cos(\omega t) + \frac{Ab\omega}{b^2\omega^2 + \left(k - m\omega^2\right)^2}\sin(\omega t).$$

For small time $t \approx 0$, we see the solution is made of the sum of two parts, a sinusoid with natural frequency ω_0 and a sinusoid with the driving frequency ω. However as $t \to \infty$, the behavior transitions to being governed entirely by a sinusoid with driving frequency. This is because:

$$\lim_{t\to\infty} e^{-\alpha t}\left[A\cos(\omega_0 t - \phi)\right] = 0.$$

We illustrate this in Fig. 23.2 where we see the initial transient behavior settles into the driven. This long term behavior is called stationary while the initial behavior is called transient or transitory.

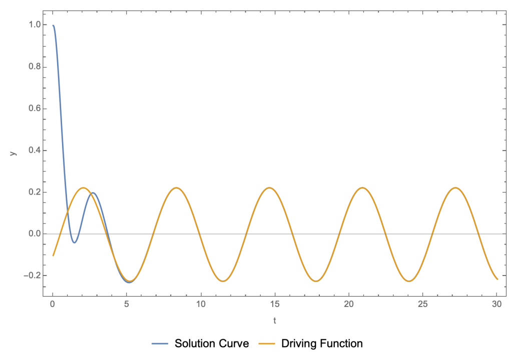


FIGURE 23.2. Damped driven oscillation goes through a transitory behavior to arive at the stationary behavior governed by a driving frequency. Here, $m = 1$, $b = 2$, $k = 5$ $A = 1$ and $\omega = 1$.

LESSON 24

1. More Examples of the Method of Undetermined Coefficients

Example 24.1. Consider the ODE:

$$y'' - y' - 2y = e^{-x}$$

Step 1: Find the roots of the characteristic equation:

$$s^2 - s - 2 = (s-2)(s+1) = 0.$$

The two roots are $s = -1$ and $s = 2$. Notice e^{-1x} appears on the right-hand-side. We must be careful here.

Step 2: Assume the structure of the solution:

$$\psi(x) = x\left(ae^{-x}\right) = axe^{-x},$$

where multiplication by x arises because -1 is a root of the characteristic equation and e^{-x} is on the right-hand-side.

Step 3: Compute the $L(\psi)$, where:

$$L = \frac{d^2}{dx^2} - \frac{d}{dx} - 2,$$

this is the operator defining the original ODE. To compute this we need:

$$\frac{d\psi}{dx} = ae^{-x} - axe^{-x}$$

$$\frac{d^2\psi}{dx^2} = -ae^{-x} - ae^{-x} + axe^{-x} = -2ae^{-x} + axe^{-x}.$$

Computing $L[\psi(x)]$ we have:

$$\frac{d^2\psi}{dx^2} - \frac{d\psi}{dx} - 2\psi(x) = -3ae^{-x}.$$

To solve the ODE we need:

$$L[\psi(x)] = -3ae^{-x} = e^{-x},$$

where the right-hand-side comes from the original ODE. To make this true, we need:

$$-3a = 1 \implies a = -\frac{1}{3}.$$

Therefore:

$$\psi(x) = -\frac{1}{3}xe^{-x}.$$

Step 4: The general solution to the homogeneous ODE:

$$L[y] = y'' - y' - 2y = 0,$$

is:

$$\varphi(x) = C_1 e^{-x} + C_2 e^{2x}.$$

Step 5: The complete solution is thus:

$$y(x) = \varphi(x) + \psi(x) = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{3} x e^{-x}.$$

Example 24.2. Consider the ODE:

$$\ddot{x} + 2\dot{x} + 5x = \sin(t)$$

Step 1: Find the roots of the characteristic equation:

$$s^2 + 2s + 5 = 0.$$

Using the quadratic formula we see:

$$s = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

Note, the right-hand-side does not look like $e^{-t}\sin(2t)$ or $e^{-t}\cos(2t)$, so we do not have to multiply by an extra power of t in $\psi(t)$.
Step 2: Assume the structure of the solution:

$$\psi(t) = a\cos(t) + b\sin(t).$$

Step 3: Compute $L(\psi)$ where:

$$L = \frac{d}{dt^2} + 2\frac{d}{dt} + 5,$$

this is the operator defining the original ODE. To compute this we need:

$$\begin{aligned} \frac{d\psi}{dt} &= -a\sin(t) + b\cos(t) \\ \frac{d^2\psi}{dt^2} &= -a\cos(t) - b\sin(t). \end{aligned}$$

Computing $L[\psi(t)]$ yields:

$$\frac{d^2\psi}{dt^2} + 2\frac{d\psi}{dt} + 5\psi = (4a + 2b)\cos(t) + (4b - 2a)\sin(t).$$

To solve the ODE, we need:

$$L[\psi(x)] = (4a + 2b)\cos(t) + (4b - 2a)\sin(t) = \sin(t),$$

where the right-hand-side comes from the original ODE. So we need a and b to satisfy:

$$\begin{aligned} 4a + 2b &= 0 \\ -2a + 4b &= 1 \end{aligned}$$

Therefore $b = \frac{1}{5}$ and $a = -\frac{1}{10}$. This means:

$$\psi(t) = -\frac{1}{10}\cos(t) + \frac{1}{5}\sin(t).$$

Step 4: The general solution to the homogeneous ODE:

$$L[x] = \ddot{x} + 2\dot{x} + 5x = 0,$$

is:

$$\varphi(t) = e^{-t}\left[C_1\cos(2t) + C_2\sin(2t)\right].$$

Step 5: The complete solution is thus:

$$x(t) = \varphi(t) + \psi(t) = e^{-t}\left[C_1\cos(2t) + C_2\sin(2t)\right] - \frac{1}{10}\cos(t) + \frac{1}{5}\sin(t).$$

Example 24.3. Consider the ODE:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = xe^{-3x}$$

Step 1: Find the roots of the characteristic equation:

$$s^2 + 6s + 9 = (s+3)(s+3) = 0.$$

There is a double root at $s = -3$. Notice e^{-3x} appears on the right-hand-side, so we need use a multiple of x^2 in the particular solution form.
Step 2: There is a first degree polynomial x and an exponential on the right-hand-side. Assume the structure of the solution:

$$\psi(x) = x^2\,(ax+b)\,e^{-3x} = (ax^3 + bx^2)e^{-3x}.$$

Step 3: Compute $L(\psi)$ where:

$$L = \frac{d}{dx^2} + 6\frac{d}{dx} + 9,$$

this is the operator defining the original ODE. To compute this we need:

$$\begin{aligned}\frac{d\psi}{dx} &= \left(3ax^2 + 2bx\right)e^{-3x} - 3(ax^3 + bx^2)e^{-3x} = \\ &\quad\left(-3ax^3 + (3a-3b)x^2 + 2bx\right)e^{-3x} \\ \frac{d^2\psi}{dx^2} &= (6ax+2b)\,e^{-3x} - 3\left(3ax^2+2bx\right) - 3\left(3ax^2+2bx\right)e^{-3x} + 9\left(ax^3+bx^2\right)e^{-3x} = \\ &\quad\left(9ax^3 + (9b-18a)x^2 + (6a-12b)x + 2b\right)e^{-3x}\end{aligned}$$

Computing $L[\psi(x)]$ yields:

$$L[\psi(x)] = (6ax + 2b)e^{-3x}$$

To solve the ODE, we need:

$$L[\psi(x)] = xe^{-3x}$$

where the right-hand-side comes from the original ODE. So we need a and b to satisfy:

$$\begin{aligned}6a &= 1 \\ 2b &= 0\end{aligned}$$

Therefore $b =$ and $a = \frac{1}{6}$. This means:

$$\psi(x) = \frac{1}{6}x^3e^{-3x}$$

Step 4: The general solution to the homogeneous ODE:

$$L[y] = y'' + 6y' + 9y = 0,$$

is:

$$\varphi(x) = C_1 e^{-3x} + C_2 x e^{-3x}.$$

Step 5: The complete solution is thus:

$$y(x) = \varphi(x) + \psi(x) = C_1 e^{-3x} + C_2 x e^{-3x} + \frac{1}{6} x^3 e^{-3x}.$$

Example 24.4. Consider the ODE:

$$\ddot{x} + 2\dot{x} + 10x = e^{-t}\sin(3t)$$

Step 1: Find the roots of the characteristic equation:

$$s^2 + 2s + 10 = 0/$$

Using the quadratic formula we see:

$$s = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i.$$

Notice that $e^{-t}\sin(3t)$ is on the right-hand-side, which we can think of as the real part of $-ie^{(-1+3i)t}$, so the root appears in right-hand-side. We need add a multiple of t to the form of $\psi(t)$.

Step 2: Assume the structure of the solution:

$$\psi(t) = t\left(a\cos(3t) + b\sin(3t)\right)e^{-t}.$$

Step 3: Compute $L(\psi)$ where:

$$L = \frac{d}{dx^2} + 6\frac{d}{dx} + 9,$$

this is the operator defining the original ODE. Computing this operator's action on $\psi(t)$ is algebraically tedious. Using a computer algebra system, we have:

$$L[\psi(t)] = 6e^{-t}\left[b\cos(3t) - a\sin(3t)\right].$$

From this we conclude that: $b = 0$, since there is no $\cos(3t)$ on the right-hand-side of the original ODE. Thus we need:

$$-6ae^{-t}\sin(3t) = e^{-t}\sin(3t),$$

which implies $a = -\frac{1}{6}$. Therefore:

$$\psi(t) = -\frac{1}{6} t e^{-t}\sin(3t).$$

Step 4: The general solution to the homogeneous ODE:

$$L[x] = \ddot{x} + 2\dot{x} + 10x = 0$$

is:

$$\varphi(t) = e^{-t}\left[C_1\cos(3t) + C_2\sin(3t)\right].$$

Step 5: The complete solution is thus:

$$x(t) = \varphi(t) + \psi(t) = e^{-t}\left[C_1\cos(3t) + C_2\sin(3t)\right] - \frac{1}{6} t e^{-t}\sin(3t).$$

Module 6

The Laplace Transform

LESSON 25

1. Function Classes

Remark 25.1. We will begin our study of the Laplace transform, a powerful technique for solving differential equations, by considering the class of functions on which they are most useful. For historical reasons all functions will be in terms of the independent variable t.

Definition 25.2 (Differentiability Classes)**.** A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is C^k if all partial derivatives of f up to order k exist and the functions resulting from differentiation are continuous. That is for all $j \leq k$:

$$v(x_1, \dots, x_n) = \frac{\partial^j f}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}} \tag{25.1}$$

exists and is continuous when $m_1 + \cdots + m_n = j$. If a function has all derivatives (and they are continuous), then the function is C^∞.

Remark 25.3. Some books define *smooth* to mean continuous and differentiable. Others require the function to be continuous and twice differentiable to be smooth some require a function to be C^∞ to be smooth

Example 25.4. The function:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is C^1. To see this, note that:

$$f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

is continuous but $f''(0)$ is not continuous and (in fact) doesn't even exist at $x = 0$. This is illustrated in Fig. 25.1.

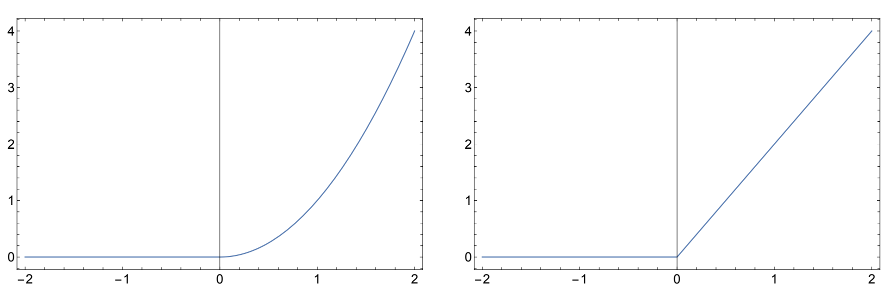


FIGURE 25.1. A C^1 function is illustrated whose derivative is C^0, that is continuous but without a continuous derivative.

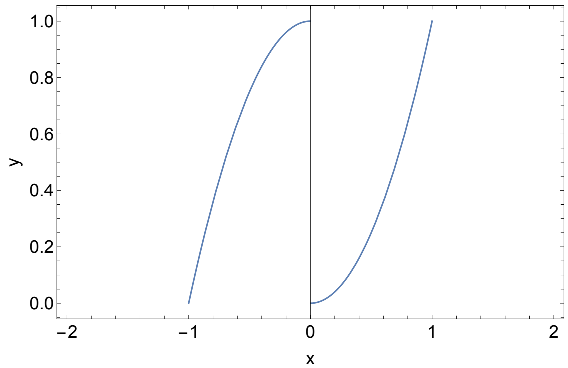


FIGURE 25.2. A piecewise C^1 function on the interval $[-1, 1]$.

Definition 25.5 (Piecewise C^1)**.** A function $f : [-L, L] \to \mathbb{R}$ is piecewise C^1 if the interval can be broken into a finite number of sub-intervals $[l_1, l_2], \ldots, [l_{n-1}, l_n]$ so that:

(1) $f(x)$ is C^1 on (l_i, l_{i+1}) for all i and
(2) $f(x)$ is left (right) continuous at l_{i+1} (l_i).

Remark 25.6. A function is piecewise C^0 if we replace C^1 in the definition by C^0. This is also called being *piecewise continuous*.

Remark 25.7. A piecewise C^1 function (sometimes called piecewise smooth, but not always) is a generally well behaved function that has a finite number of jump discontinuities. This eliminates functions with cusps, asymptotes, vertical derivatives or generally anything non-physical. The function:

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -1 \leq x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$$

We illustrate this function in Fig. 25.2.

Remark 25.8. A piecewise continuous function can be integrated by breaking the integral into the intervals on which the function is continuous and adding the results together.

Definition 25.9 (Exponential Type)**.** A function $f(t)$ is said to be *of exponential type with order* α if there are constants $T > 0$, $M > 0$ and a value α so that:

$$|f(t)| \leq M \exp(\alpha|t|).$$

for all $t > T$.

Remark 25.10. A function is of exponential type if its growth is bounded by an exponential function. In particular, this just means it cannot grow too large too fast.

2. The Laplace Transform

Definition 25.11 (Laplace Transform)**.** Let $f(t)$ be a function. Then the *Laplace Transform* of f is:

$$\mathcal{L}[f](s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt, \tag{25.2}$$

assuming that the integral exists.

Remark 25.12. In the strictest sense $F(s)$ is a complex function of a complex variable s. However, we will not use this feature of the transform in our work (except once).

Example 25.13. Consider the simple function $f(t) = c$, where c is a constant. We can find its Laplace transform as:

$$\mathcal{L}\left[c\right] = \int_0^\infty ce^{-st}\,dt = -\frac{c}{s}e^{-st}\Big|_0^\infty = \lim_{t\to\infty}\frac{c}{s}e^{-st} - \frac{c}{s} = \frac{c}{s}.$$

Example 25.14. Consider the function e^{at}. We can find its Laplace transform as:

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{(a-s)t} = \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty = \lim_{t\to\infty}\frac{1}{a-s}e^{(a-s)t} - \frac{1}{a-s}.$$

This integral is defined *just in case* $s > a$ (or in the case we treat s as complex, $\mathrm{Re}(s) > a$). In this case we see:

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \quad \text{if } s > a.$$

THEOREM 25.15. *Suppose $f(t)$ is piecewise continuous and exponential of order α. Then $F(s)$ exists assuming $s > \alpha$.*

PROOF. Let T be the time constant in Definition 25.9.

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt = \int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st}f(t)\,dt.$$

The first integral clearly converges. For the second integral we have:

$$\int_T^\infty e^{-st}f(t)\,dt \leq \int_T^\infty \left|e^{-st}f(t)\right|dt \leq$$
$$\int_T^\infty e^{-st}Me^{\alpha t}\,dt = \int_T^\infty Me^{(\alpha-s)t}\,dt = \frac{M}{\alpha-s}e^{(\alpha-s)t}\Big|_T^\infty = \frac{M}{s-\alpha}e^{(\alpha-s)T},$$

when we assume $s > \alpha$. Therefore, this integral converges and the Laplace transform exists just in case $s > \alpha$. $\square$

Remark 25.16. The Laplace transform exists for most of the functions we will worry about in this class.

LESSON 26

1. Other Properties of the Laplace Transform

THEOREM 26.1. *The Laplace transform is linear.*

PROOF. Let $a \in \mathbb{R}$. We see:

$$\mathcal{L}[af(t) + g(t)] = \int_0^\infty [af(t) + g(t)]\, e^{-st}\, dt =$$
$$a\int_0^\infty f(t)e^{-st}\, dt + \int_0^\infty g(t)e^{-st}\, dt = a\mathcal{L}[f(t)] + \mathcal{L}[g(t)].$$

□

Example 26.2. Recall the Laplace transform is a complex function of a complex variable s, which we are largely ignoring. However, we can use this fact along with Example 25.14 and Theorem 26.1 to conveniently compute $\mathcal{L}[\sin(\omega t)]$. Recall from Euler's theorem: $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Using this information we can write:

$$\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right).$$

Now applying the Laplace transform we have:

$$\mathcal{L}[\sin(\omega t)] = \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right).$$

Simplifying the fraction we have:

$$\mathcal{L}[\sin(\omega t)] = \frac{1}{2i}\frac{2i\omega}{s^2 + \omega^2} = \frac{\omega}{s^2 + \omega^2}.$$

Derivation 26.3 (Exponential Multiplication)**.** Consider the following Laplace transform:

$$\mathcal{L}\left[f(t)e^{at}\right] = \int_0^\infty f(t)e^{at}e^{-st}\, dt = \int_0^\infty f(t)e^{-(s-a)t}\, dt$$

Let $\sigma = s - a$. Then:

$$\int_0^\infty f(t)e^{-(s-a)t}\, dt = \int_0^\infty f(t)e^{-\sigma t}\, dt = \mathcal{L}[f(t)](\sigma) = F(\sigma) = F(s-a).$$

We have proved a theorem.

THEOREM 26.4. *If $f(t)$ is a function with Laplace transform $F(s)$, then:*

$$\mathcal{L}\left[f(t)e^{at}\right] = F(s-a).$$

□

Remark 26.5. Thus we have proved that multiplying a function by e^{at} shifts its Laplace transform to the right.

Derivation 26.6 (Derivatives and Laplace Transforms)**.** Assume $f(t)$ is differentiable with piecewise continuous derivative and of exponential type. Consider the Laplace transform:

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}\,dt.$$

Let's proceed by integration by parts. Let $u(t) = e^{-st}$ and $dv = f'(t)dt$. Then: $du = -se^{-st}$ and $v = f(t)$. We have:

$$f(t)e^{-st}\Big|_0^\infty - \int_0^\infty -f(t)se^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + s\mathcal{L}[f(t)].$$

We know that $|f(t)| \leq Me^{\alpha t}$. Therefore if $s > \alpha$ we have:

$$f(t)e^{-st}\Big|_0^\infty = \lim_{t\to\infty} f(t)e^{-st} - f(0).$$

No:

$$\lim_{t\to\infty} |f(t)e^{-st}| \leq \lim_{t\to\infty} Me^{\alpha t}e^{-st} = \lim_{t\to\infty} Me^{(\alpha-s)t} = 0,$$

when we assume $s > \alpha$. Therefore:

$$f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = -f(0) + s\mathcal{L}[f(t)].$$

Consequently we have proved a theorem.

Theorem 26.7. *If $f(t)$ is differentiable with piecewise continuous derivative and of exponential type with order α, then if $s > \alpha$:*

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

□

Example 26.8. Using Theorem 26.7 we can compute $\mathcal{L}[\cos(\omega t)]$. Notice:

$$\frac{d}{dt}\left[\frac{1}{\omega}\sin(\omega t)\right] = \cos(\omega t).$$

So:

$$\mathcal{L}[\cos(\omega t)] = \mathcal{L}\left\{\frac{d}{dt}\left[\frac{1}{\omega}\sin(\omega t)\right]\right\} = \frac{1}{\omega}\left\{s\mathcal{L}[\sin(\omega t)] + \sin(0)\right\} =$$

$$\frac{s}{\omega}\frac{\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^a}.$$

Remark 26.9. The following corollary can be proved by induction.

Corollary 26.10. *If $f(t)$ has n derivatives, with the n^th derivative being piecewise continuous and all of exponential order α. Then if $s > \alpha$ we have:*

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

In particular we have:

$$\mathcal{L}\left[f''(t)\right] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

□

Derivation 26.11. Consider the question: Is there a $g(t)$ so that if $\mathcal{L}[f(t)] = F(s)$, then $F'(s) = \mathcal{L}[g(t)]$. Then:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Taking the derivative with respect to s we have:

$$F'(s) = \frac{d}{ds}\left[\int_0^\infty f(t)e^{-st}\,dt\right].$$

Passing the derivative through yields:

$$F'(s) = \left[\int_0^\infty -tf(t)e^{-st}\,dt\right].$$

Thus we conclude that if $f(t)$ is piecewise continuous and of exponential type (implying that $tf(t)$ is piecewise continuous and of exponential type) that:

$$\mathcal{L}[-tf(t)] = F'(s).$$

Repeating this argument we can generalize the result.

THEOREM 26.12. *Suppose that $f(t)$ is piecewise continuous and of exponential type. Then:*

$$F^{(n)}(s) = \mathcal{L}\left[(-1)^n t^n f(t)\right]$$

or equivalently:

$$(-1)^n F^{(n)}(s) = \mathcal{L}\left[t^n f(t)\right].$$

□

2. Solving ODE's with the Laplace Transform

Remark 26.13. The Laplace transform is invertible with formula:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-Ti}^{\gamma+Ti} F(s)e^{st}\,ds,$$

where $\gamma \in \mathbb{R}$ is chosen to ensure convergence of $F(s)$. Using this integral is <u>far</u> outside the scope of this course (since it requires contour integration). Consequently, we will use the simplification and inspection method of inversion. However, the following theorem should seem reasonable (and is useful).

THEOREM 26.14. *The inverse Laplace transform is linear.* □.

Example 26.15. Suppose:

$$F(s) = \frac{3}{s^2-1}.$$

We can write this as:

$$\frac{3}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right).$$

By linearity, we can write:

$$\mathcal{L}^{-1}[F(s)] = \frac{3}{2}\left[\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\right].$$

We know from Example 25.14 that:

$$\frac{1}{s-1} = \mathcal{L}\left(e^{t}\right)$$
$$\frac{1}{s+1} = \mathcal{L}\left(e^{-t}\right)$$

So we can conclude that:

$$\mathcal{L}^{-1}[F(s)] = \frac{3}{2}\left(e^{t} - e^{-t}\right) = 3\sinh(t).$$

Example 26.16. Let's resolve the following ODE:

$$\ddot{y} + \omega_0^2 y = \sin(\omega t) \qquad y(0) = 0, \dot{y}(0) = 0$$

using a Laplace transform. Taking the Laplace transform of both sides yields:

$$s^2 Y(s) + sy(0) - \dot{y}(0) + \omega_0 Y(s) = s^2 Y(s) + \omega_0 Y(s) = \frac{\omega}{s^2 + \omega^2}.$$

We can now factor the left-hand-side to see:

$$(s^2 + \omega_0^2)Y(s) = \frac{\omega}{s^2 + \omega^2}$$

Dividing we have:

$$Y(s) = \frac{\omega}{(s^2 + \omega^2)(s^2 + \omega_0^2)}.$$

Now all that remains is to find the inverse Laplace transform. Let's find A and B so that:

$$\frac{1}{(s^2 + \omega^2)(s^2 + \omega_0^2)} = \frac{A}{s^2 + \omega^2} + \frac{B}{s^2 + \omega_0^2}.$$

We can worry about the ω multiplier later, since the Laplace transform and its inverse are linear. Using the method of partial fractions yields the equations:

$$A + B = 0$$
$$\omega_0^2 A + \omega^2 B = 1$$

Some algebra shows:

$$A = \frac{1}{\omega_0^2 - \omega^2} \qquad B = \frac{1}{\omega^2 - \omega_0^2}.$$

For simplicity we will substitute these values at the end. We now have:

$$Y(s) = \frac{\omega A}{s^2 + \omega^2} + \frac{\omega B}{s^2 + \omega_0^2}.$$

We know:

$$\mathcal{L}^{-1}\left[\frac{\omega A}{(s^2 + \omega^2)}\right] = A\sin(\omega t)$$
$$\mathcal{L}^{-1}\left[\frac{\omega B}{(s^2 + \omega_0^2)}\right] = \frac{\omega B}{\omega_0}\sin(\omega_0 t).$$

Substituting the inverse Laplace transforms and A and B we conclude:

$$y(t) = \frac{\omega_0 \sin(t\omega) - \omega \sin\left(t\omega_0\right)}{\omega_0({}_0^2 - \omega^2)},$$

which has the same form as the solution in Example 22.8 except here we used a forcing function with $\sin(\omega t)$.

Remark 26.17. This is the power of the Laplace transform. It turns differential equations into algebraic equations, which can then be manipulated more easily. The true power of this approach, however, comes in allowing us to solve a new class of differential equations - those with discontinuous right hand sides.

Remark 26.18. Notice we needed to use a partial fraction method to separate the terms to "see" how to invert the Laplace transform. This is a hallmark of this approach (when one does not use a computer to invert the Laplace transform). Going forward, we will assume some familiarity with the method of partial fractions.

LESSON 27

1. The Dirac Delta and Unit Step Functions

Remark 27.1. There are several ways to formally define the Dirac delta function. We'll use the limit of functions approach and hint at the linear functional method. It is also worth noting there are many ways to approach the limit of functions definition. We'll use the one from [Olv14] because it's pretty and uses a fact from Calculus 1.

Lemma 27.2. *The following equation holds:*

$$\int \frac{n}{\pi(1+n^2x^2)}\,dx = \frac{1}{\pi}\tan^{-1}(nx).$$

□

Definition 27.3. Let:

$$f_n(x) = \frac{n}{\pi(1+n^2x^2)}.$$

Then:

$$\delta(x) = \lim_{n\to\infty} f_n(x).$$

Remark 27.4. The functions $f_n(x)$ are illustrated in Fig. 27.1.

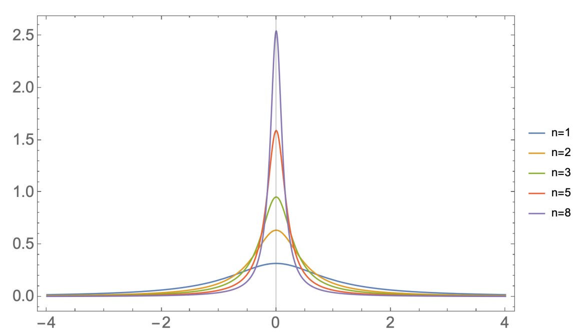


FIGURE 27.1. The limit of the functions $f_n(x)$ converges to what we call the Dirac Delta function.

Corollary 27.5. *The following equation holds:*

$$\int_{-\infty}^{\infty} \frac{n}{\pi(1+n^2x^2)}\,dx = 1.$$

Proof. We are assuming n is a positive integer. We know:

$$\lim_{x\to\infty} \frac{1}{\pi}\tan^{-1}(nx) = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$
$$\lim_{x\to-\infty} \frac{1}{\pi}\tan^{-1}(nx) = -\frac{1}{\pi}\frac{\pi}{2} = -\frac{1}{2}$$

The result follows immediately. □

Remark 27.6. Given Corollary 27.5 we would *hope* that:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty}\left[\lim_{n\to\infty} f_n(x)\right]\,dx = \int_{-\infty}^{\infty}\delta(x)\,dx = 1.$$

As we've discussed in the past, showing that the limit can be passed through the integral requires showing convergence properties that are well outside the scope of this course. More to the point, with ordinary Riemann (or even Lebesgue) integration, this equality is **incorrect**. To make this sensible mathematically requires an entirely different course on analysis. We are just going to accept it as true for expediency. More to the point we're actually going to accept the following theorem.

Theorem 27.7. *Suppose* $\xi \in [a,b]$*. Then:*

$$\int_a^b \delta(x-\xi)\,dx = 1.$$

On the other hand, if $\xi \notin [a,b]$*, then:*

$$\int_a^b \delta(x-\xi)\,dx = 0.$$

□

Derivation 27.8 (Heaviside Step Function)**.** Consider the function:

$$H(x) = \int_{-\infty}^{x}\delta(s)\,ds.$$

We see immediately from Theorem 27.7 and that this is the step function:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

This function is not defined at $x = 0$. This is the *Heaviside step function* or *unit step function*. As a consequence of this result we say:

$$\frac{dH}{dx} = \delta(x).$$

The Heaviside step function is shown in Fig. 27.2. For many applications, the value at $x = 0$ is filled in with $H(0) = \frac{1}{2}$ and for other applications we assume $H(0) = 0$. This does not really affect the relation to $\delta(x)$.

Derivation 27.9. Assume $g(x)$ is a function defined on $[a,b]$ with $\xi \in (a,b)$. Another interesting property of $f_n(x)$ is:

$$\lim_{n\to\infty}\int_a^b f_n(x-\xi)g(x)\,dx = g(\xi).$$

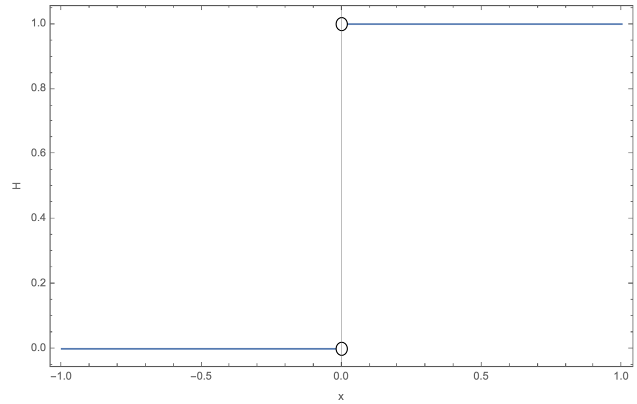


FIGURE 27.2. The Heaviside step function.

In much the same way we did with Eq. (27.1), we can assert that if $\xi \in [a, b]$, then:

$$\int_a^b \delta(x-\xi)g(x)\,dx = g(\xi). \tag{27.1}$$

Interestingly, because the δ function is even we also have:

$$\int_a^b \delta(x-s)g(s)\,dx = g(x). \tag{27.2}$$

We now have the following theorem, which we certainly will not prove but will take on faith.

THEOREM 27.10. *If $\xi \in [a, b]$ and $g(x)$ is continuous at ξ, then:*

$$\int_a^b \delta(x-\xi)g(x)\,dx = g(\xi).$$

□

Derivation 27.11. We can now compute the Laplace transform of the Dirac function and the Heaviside step function. Assume $\xi \in [0, \infty)$:

$$\mathcal{L}[\delta(t-\xi)] = \int_0^\infty \delta(t-\xi)e^{-st}\,dt$$

Applying Eq. (27.1) we have:

$$\mathcal{L}[\delta(t-\xi)] = e^{-s\xi}.$$

Likewise, if we compute:

$$\mathcal{L}[H(t-\xi)] = \int_0^\infty H(t-\xi)e^{-st}\,dt = \int_\xi^\infty e^{-st}\,dt = -\left.\frac{e^{-st}}{s}\right|_\xi^\infty = \frac{e^{-\xi s}}{s}$$

We can generalize this.

Remark 27.12. To shift a function $f(t)$ to the right by ξ time-units, one simply uses input $t-\xi$. That is $f(t-\xi)$ shifts $f(t)$ to the right. However, if $f(t)$ is specialized to model only positive time ($t \geq 0$) this shift has the unfortunate affect of adding information at $t = 0$. To cancel this information, we can multiply by the shifted Heaviside step function: $H(t-\xi)f(t-\xi)$.

Theorem 27.13. *Suppose $\xi \in [0, \infty)$, then:*
$$\mathcal{L}[H(t-\xi)f(t-\xi)] = e^{-\xi s}F(s)$$

Proof. Compute:
$$\mathcal{L}[H(t-\xi)f(t-\xi)] = \int_0^\infty H(t-\xi)f(t-\xi)e^{-st}\,dt.$$
Let $\tau = t - \xi$ and $d\tau = dt$. Then we have:
$$\mathcal{L}[H(t-\xi)f(t-\xi)] = \int_{-\xi}^\infty H(\tau)f(\tau)e^{-s(\tau+\xi)}\,d\tau = \int_{-\xi}^\infty H(\tau)f(\tau)e^{-s\tau}e^{-s\xi}\,d\tau = e^{-s\xi}\int_{-\xi}^\infty H(\tau)f(\tau)e^{-s\tau}\,d\tau.$$
But $H(\tau) = 0$ if $\tau < 0$ and otherwise $H(\tau) = 1$ if $\tau > 0$. Consequently the final integral becomes:
$$e^{-s\xi}\int_{-\xi}^\infty H(\tau)f(\tau)e^{-s\tau}\,d\tau = e^{-s\xi}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-s\xi}\mathcal{L}[f(\tau)] = e^{-s\xi}F(s).$$
□

LESSON 28

1. Solving More ODE's with the Laplace Transform

Example 28.1. Consider the following differential equation:

$$\frac{dy}{dt} - ky = \delta(t - \xi),$$

where $k \in \mathbb{R}$ is a constant. Assume the initial condition $y(0) = 0$. We can now solve this problem. For simplicity, we assume $\xi \geq 0$. To solve this problem, apply the Laplace transform to both sides:

$$\mathcal{L}\left(\frac{dy}{dt}\right) = \mathcal{L}\left[ky + \delta(t - \xi)\right].$$

Using Theorem 26.7, we have:

$$sY(s) - y(0) = kW(s) + e^{-\xi s}.$$

We know $y(0) = 0$ and we can solve for $Y(s)$ to obtain:

$$Y(s) = \frac{e^{-\xi s}}{s - k} = e^{-\xi s}\frac{1}{s - k}.$$

Now we can back infer that:

$$y(t) = H(t - \xi)e^{k(t-\xi)} = \begin{cases} e^{k(t-\xi)} & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

For consistency with the initial condition, when $\xi = 0$ we write:

$$y(t) = H(t)e^{kt} = \begin{cases} e^{kt} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Example 28.2 (Non-homogeneous Piecewise Continuous Terms - Part 1)**.** Let's solve the problem:

$$y'' + 3y' + 2y = \left[H(t - 5) - H(t - 20)\right] e^t \qquad y(0) = y'(0) = 0.$$

Using the definition of the Heaviside function, we can see that:

$$H(t - 5) - H(t - 20) = \begin{cases} 1 & 5 < t < 20 \\ 0 & \text{otherwise.} \end{cases}$$

This models a switch that turns off at time $t = 5$ and turns off at time $t = 20$. Let's apply the Laplace transform to $H(t - \xi)e^{at}$. We know from Derivation 27.11:

$$\mathcal{L}[H(t - \xi)] = \frac{e^{-\xi s}}{s}.$$

We know from Theorem 26.4 we have:

$$\mathcal{L}\left[f(t)e^{at}\right] = F(s-a).$$

Putting these two things together we see:

$$\mathcal{L}[H(t-\xi)e^{at}] = \mathcal{L}[H(t-\xi)](s-a) = \left.\frac{e^{-\xi s}}{s}\right|_{s=s-a} = \frac{e^{-\xi(s-a)}}{s-a}. \tag{28.1}$$

Now taking the Laplace transform of the original equation we have:

$$s^2Y(s) - sy'(0) - y(0) + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{e^{-5(s-1)}}{s-1} - \frac{e^{-20(s-1)}}{s-1}.$$

Simplifying we have:

$$Y(s)\left(s^2+3s+2\right) = \frac{e^{-5(s-1)} - e^{-20(s-1)}}{s+1}$$

Dividing through and factoring yields:

$$Y(s) = \frac{e^{-5(s-1)} - e^{-20(s-1)}}{(s-1)(s+1)(s-2)}$$

Now apply partial fraction decomposition:

$$Y(s) = \left(e^{-5(s-1)} - e^{-20(s-1)}\right)\left[\frac{1}{6(s+1)} + \frac{1}{3(s-2)} - \frac{1}{2(s-1)}\right].$$

Parts of this are now easy to invert and other parts are difficult. We can read from Eq. (28.1) that:

$$\mathcal{L}^{-1}\left[-\frac{e^{-5(s-1)} - e^{-20(s-1)}}{2(s-1)}\right] = -\frac{\mathcal{H}(t-5)e^t - \mathcal{H}(t-20)e^t}{2}.$$

The more difficult part are the Laplace transforms:

$$\mathcal{L}^{-1}\left[\frac{e^{-5(s-1)} - e^{-20(s-1)}}{3(s-2)}\right]$$

and

$$\mathcal{L}^{-1}\left[\frac{e^{-5(s-1)} - e^{-20(s-1)}}{6(s+1)}\right].$$

To complete the example, we need a derivation.

Derivation 28.3. Consider the Laplace transform of $e^{a(t-c)}f(t)$. We have:

$$\mathcal{L}\left[e^{a(t-c)}f(t)\right] = \int_0^\infty e^{a(t-c)}f(t)e^{-st}\,dt = \int_0^\infty e^{-(s-a}f(t)e^{-ac}\,dt = e^{-ac}F(s-a),$$

using the same reasoning as in Theorem 26.4.

Example 28.4 (Non-homogeneous Piecewise Continuous Terms - Part 2)**.** We now compute:

$$\mathcal{L}^{-1}\left[\frac{e^{-5(s-1)}}{(s+1)}\right].$$

From the denominator, we know we are dealing with a function of form $e^{-(t-c)}f(t)$ because the $(s+1) = s-(-1)$ in the denominator implies some multiplier of form $e^{-(t-c)}$. From the numerator we know that $f(t) = H(t-5)$. We must identify c; we already know $a = -1$. Using Derivation 28.3 and Eq. (28.1) we know:

$$\mathcal{L}\left[e^{-(t-c)}H(t-5)\right] = e^c \frac{e^{-5(s+1)}}{s+1} \underbrace{=}_{\text{want}} \frac{e^{-5(s-1)}}{(s+1)}$$

Relating the numerator of the computed Laplace transform to the numerator of the known Laplace transform we have:

$$e^c e^{-5(s+1)} = e^{-5(s-1)}.$$

This implies:

$$c - 5s - 5 = -5s + 5 \implies c = 10.$$

Thus we have deduced that:

$$\mathcal{L}^{-1}\left[-\frac{e^{-5(s-1)}}{(s+1)}\right] = e^{-(t-10)}H(t-5).$$

We go through the same process to find the inverse transform of the right-hand-side and we deduce:

$$y(t) = \frac{1}{6}\left[e^{10-t}H(t-5) - e^{40-t}H(t-20)\right] + \\ \frac{1}{3}\left[e^{2(t-5)+5}H(t-5) - e^{2(t-20)+20}H(t-20)\right] - \frac{1}{2}\left[e^t H(t-5) - e^t H(t-20)\right].$$

This can be factored as:

$$y(t) = H(t-5)\left[\frac{1}{6}e^{10-t} + \frac{1}{3}e^{2(t-5)+5} - \frac{1}{2}e^t\right] - H(t-20)\left[\frac{1}{6}e^{40-t} + \frac{1}{3}e^{2(t-20)+20} - \frac{1}{2}e^t\right].$$

2. Introduction to Green's Functions*

Definition 28.5 (Green's Function)**.** Consider a one-dimensional linear operator L with some given boundary conditions. We assume the operator works on the variable t; e.g., L is a derivative with respect to t. A *Green's function solution* is a function $G(t,s)$ satisfying the boundary conditions with the property that:

$$L[G(t,s)] = \delta(t-s).$$

Note: I'm using the preferred Physics definition of the Green's function. In math, it's defined as $L[G(t,s)] = \delta(s-t)$. As we know from Eq. (27.2), it makes no difference.

Remark 28.6. We first note that not every operator admits a Green's function solution. Second, if the operator is translation invariant (e.g., has constant coefficients), then the Green's function is a univariate function $G(z)$ and:

$$G(t,s) = G(t-s).$$

Derivation 28.7. Assume an operator L in one dimension admits a Green's function. It's worth asking, why would anyone care? Consider the following non-homogenous (and therefore hard) problem:

$$L[w(t)] = f(t),$$

where $w(t)$ is an unknown function satisfying some boundary conditions on an interval $[a, b]$, with $0 \in (a, b)$. (If zero is not in the interval, we can always shift the interval to make this true.) Suppose $G(t, s)$ is a Green's function. Then:

$$L[G(t, s)] = \delta(t - s).$$

Now consider the integral on $[a, b]$:

$$\int_a^b G(t, s)f(s)\, ds.$$

Apply the operator L and assume that L commutes with the integral (a big assumption).

$$L\left[\int_a^b G(t, s)f(s)\, ds\right] = \int_a^b L[G(t, s)]f(s)\, ds = \int_a^b \delta(t - s)f(s)\, ds = f(t),$$

for $t \in [a, b]$. Thus the Green's function can be used to solve the non-homogenous problem by the integral operator:

$$w(t) = I[f(s)] = \int_a^b G(t, s)f(s)\, ds.$$

When $G(t, s) = G(t - s)$, then this integral is called a *convolution* of G and f and G is sometimes called the kernel. We have:

$$w(t) = \int_a^b G(t - s)f(s)\, ds.$$

Remark 28.8. The Laplace transform (and its cousin the Fourier transform) have interesting properties with respect to convolution. In particular Laplace and Fourier transforms turn convolution into multiplication. This is a topic that should be covered in Math 411 or Math 412.

Example 28.9. Using Example 28.1, we have deduced that:

$$G(t, \xi) = \begin{cases} e^{k(t-\xi)} & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

We can conclude that given the problem:

$$\begin{aligned} &\frac{dw}{dt} - kw = f(t) \\ &w(0) = 0, \end{aligned}$$

we have:

$$w(t) = \int_0^t G(t, \xi)f(\xi)\, d\xi = e^{kt}\int_0^t e^{-k\xi}f(\xi)\, d\xi.$$

LESSON 29

1. Four More Examples of the Laplace Transform

Remark 29.1. For these examples, we will be using the table of Laplace transforms available at: https://tutorial.math.lamar.edu/classes/de/Laplace_Table.aspx.

Example 29.2. To solve the problem:

$$\ddot{y} + 6\dot{y} + 5y = 0 \quad y(0) = 4, \dot{y}(0) = 0,$$

we compute the Laplace transform of both sides of the equation. We have:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 6\left[sY(s) - y(0)\right] + 5Y(s) = 0.$$

Substituting initial condition values and simplifying yields:

$$(s^2 + 6s + 5)Y(s) - 4(s + 6) - 0 = 0.$$

Isolating $Y(s)$ yields:

$$Y(s) = 4\frac{s+6}{s^2+6s+5} = 4\frac{s+6}{(s+5)(s+1)}.$$

Now apply the method of partial fractions:

$$\frac{s+6}{(s+5)(s+1)} = \frac{A}{s+1} + \frac{B}{s+5}.$$

This yields:

$$As + 5A + Bs + B = s + 6.$$

This gives the two equations:

$$A + B = 1$$
$$5A + B = 6.$$

Solving gives $A = \frac{5}{4}$ and $B = -\frac{1}{4}$. We can now write:

$$Y(s) = 4\left[\frac{5}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s+5}\right] = 5\frac{1}{s+1} - \frac{1}{s+5}.$$

Recall:

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a},$$

so we conclude:

$$y(t) = 5e^{-t} - e^{-5t}.$$

Example 29.3. To solve the forced harmonic oscillator problem:

$$\ddot{y} + 4\dot{y} = \cos(2t) \quad y(0) = \dot{y}(0) = 0$$

we compute the Laplace transform of both sides of the equation. We have:

$$s^2 Y(s) - sy(0) - \dot{y}(0) + 4Y(s) = (s^2+4)Y(s) = \frac{s}{s^2+4},$$

where the right-hand-side is the Laplace transform of $\cos(2t)$. Then we have:

$$Y(s) = \frac{s}{(s^2+4)^2}.$$

We know:

$$\mathcal{L}\left[t\sin(at)\right] = \frac{2as}{(s^2+a^2)^2},$$

setting $a = 2$ we would have:

$$\mathcal{L}\left[t\sin(2t)\right] = \frac{4s}{(s^2+4)^2} = 4\frac{s}{(s^2+4)^2}.$$

Therefore:

$$\frac{1}{4}\mathcal{L}\left[t\sin(2t)\right] = \frac{s}{(s^2+4)^2}.$$

We conclude:

$$y(t) = \frac{1}{4}t\sin(2t).$$

Example 29.4. To solve the problem:

$$\ddot{y} + 3\dot{y} + 2y = e^t \quad y(0) = 1, \dot{y}(0) = 3,$$

we compute the Laplace transform of both sides of the equation. We have:

$$s^2 Y(s) - sy(0) - \dot{y}(0) + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{1}{s-1}.$$

Simplifying and substituting in the initial conditions yields:

$$(s^2+3s+2)Y(s) - (s+3) + 3 = \frac{1}{s-1},$$

or

$$Y(s) = \frac{1}{(s-1)(s^2+3s+2)} + \frac{s}{s^2+3s+2}.$$

We can break this into pieces and apply partial fractions. First note: $s^2+3s+2 = (s+1)(s+2)$. We start with the simpler piece:

$$\frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Then: $As + 2A + Bs + B = s$. Solving for A and B yields $A = -1$, $B = 2$ or:

$$\frac{s}{(s+1)(s+2)} = \frac{2}{s+2} - \frac{1}{s+1}.$$

Now:

$$\frac{1}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

The numerator can be computed as:

$$(A+B+C)s^2 + (3A+B)s + (2A-2B-C) = 1$$

Then:

$$\begin{aligned} &A+B+C = 0 \\ &3A+B = 0 \\ &2A-2B-C = 0 \end{aligned}$$

We see $A = \frac{1}{6}$, $B = -\frac{1}{2}$ and $C = \frac{1}{3}$. Adding the two partial fractions results together yields:

$$Y(s) = \frac{1}{6}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+2} + \frac{2}{s+2} - \frac{1}{s+1} = \\ \frac{1}{6}\frac{1}{s-1} - \frac{3}{2}\frac{1}{s+1} + \frac{7}{3}\frac{1}{s+2}.$$

We conclude:

$$y(t) = \frac{1}{6}e^t - \frac{3}{2}e^{-t} + \frac{7}{3}e^{-2t}.$$

Example 29.5. Consider a damped harmonic oscillator with an impulse (hammer striking a damped harmonic oscillator):

$$\ddot{y} + 2\dot{y} + 2y = \delta(t-1) \quad y(0) = \dot{y} = 0.$$

We compute the Laplace transform of both sides of the equation:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2\left[sY(s) - y(0)\right] + 2Y(s) = e^{-s},$$

which simplifies to:

$$(s^2 + 2s + 2)Y(s) = e^{-s}.$$

Solving for $Y(s)$ yields:

$$Y(s) = e^{-s}\frac{1}{s^2+2s+2}.$$

Complete the square in the denominator to see:

$$Y(s) = e^{-s}\frac{1}{(s+1)^2+1}.$$

We now:

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2+b^2}.$$

Letting $a = -1$ and $b = 1$ we see:

$$\mathcal{L}\left[e^{-t}\sin(t)\right] = \frac{1}{(s+1)^2+1}.$$

We also know that:

$$\mathcal{L}\left[H(t-c)f(t-c)\right] = e^{-cs}F(t).$$

From this we conclude that:

$$\mathcal{L}\left[H(t-1)e^{-t-1}\sin(t-1)\right] = e^{-s}\frac{1}{(s+1)^2+1}.$$

Therefore:

$$y(t) = H(t-1)e^{-t-1}\sin(t-1) = \begin{cases} e^{t-1}\sin(t-1) & t > 1 \\ 0 & t < 0 \end{cases}$$

Module 7

Systems of Differential Equations

LESSON 30

1. Matrices

Remark 30.1. For a really good introduction to Linear Algebra see [Lan12] by Lang. Lang really does write the best books.

Definition 30.2 (Matrix)**.** An $m \times n$ matrix is a rectangular array of values (*scalars*), drawn from a set of numbers called a field. For us, the field will always be $\mathbb{R}$. We write $\mathbb{R}^{m\times n}$ to denote the set of $m \times n$ matrices with entries drawn from $\mathbb{R}$.

Example 30.3. The following are matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remark 30.4. If $\mathbf{A} \in \mathbb{R}^{m\times n}$, then the matrix consists of m rows and n columns. The element in the i^{th} row and j^{th} column of $\mathbf{A}$ is written as $\mathbf{A}_{ij}$. The j^{th} column of $\mathbf{A}$ can be written as $\mathbf{A}_{\cdot j}$, where the $\cdot$ is interpreted as ranging over every value of i (from 1 to m). Similarly, the i^{th} row of $\mathbf{A}$ can be written as $\mathbf{A}_{i\cdot}$. When $m = n$, then the matrix $\mathbf{A}$ is called *square*.

Remark 30.5 (Matrix Addition)**.** Matrices add componentwise as illustrated in the example below.

Example 30.6. Consider the sum:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 11 \end{bmatrix}$$

Definition 30.7 (Row/Column Vector)**.** A $1 \times n$ matrix is called a *row vector*, and a $m \times 1$ matrix is called a *column vector*. For the remainder of these notes, every vector will be thought of **column vector** unless otherwise noted. To save space, column vectors are written horizontally with angle brackets, such as: $\mathbf{x} = \langle x_1, \dots, x_n \rangle$.

Remark 30.8. It should be clear that any row of matrix $\mathbf{A}$ could be considered a row vector in $\mathbb{R}^n$ and any column of $\mathbf{A}$ could be considered a column vector in $\mathbb{R}^m$.

Definition 30.9 (Dot Product)**.** Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors (either row or column) with n elements. Then the *dot* product of x with y is:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{30.1}$$

Definition 30.10. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$. (Here 0 is the zero in the field over which the vectors are defined.)

Definition 30.11 (Norm)**.** If $\mathbf{x}$ is a vector, then $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ is the *norm* of the vector. It is simply the length of the vector in Euclidean space.

Definition 30.12 (Orthonormal Vectors)**.** If two vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal and both vectors have norm equal to 1, then they are said to be *orthonormal*.

Definition 30.13 (Matrix Multiplication)**.** If $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, then $\mathbf{C} = \mathbf{AB}$ is the *matrix product* of $\mathbf{A}$ and $\mathbf{B}$ and

$$\mathbf{C}_{ij} = \mathbf{A}_{i\cdot} \cdot \mathbf{B}_{\cdot j} \tag{30.2}$$

Note, $\mathbf{A}_{i\cdot} \in \mathbb{R}^{1 \times n}$ (an n-dimensional vector) and $\mathbf{B}_{\cdot j} \in \mathbb{R}^{n \times 1}$ (another n-dimensional vector), thus making the dot product meaningful.

Example 30.14.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1(5)+2(7) & 1(6)+2(8) \\ 3(5)+4(7) & 3(6)+4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \tag{30.3}$$

2. Special Matrices and Vectors

Definition 30.15 (Identity Matrix)**.** The $n \times n$ *identify matrix* is:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{30.4}$$

Example 30.16. Notice that identify matrices have a special property. If $\mathbf{B} \in \mathbb{R}^{m \times n}$ and $\mathbf{I}_m$ is the $m \times m$ identity matrix, then $\mathbf{I}_m\mathbf{B} = \mathbf{B}$. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Definition 30.17 (Zero Matrix)**.** The $n \times n$ *zero* matrix an $n \times n$ consisting entirely of 0.

Definition 30.18 (Invertible Matrix)**.** Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix. If there is a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n \tag{30.5}$$

then matrix $\mathbf{A}$ is said to be *invertible* (or *nonsingular*) and $\mathbf{A}^{-1}$ is called its *inverse*. If $\mathbf{A}$ is not invertible, it is called a *singular* matrix.

Remark 30.19. Finding a matrix inverse (assuming it exists) requires using some numerical technique. There is a formula for 2×2 matrices, which are the ones we will be using for practical problems. If:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the inverse is:

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

assuming $ad - bc \neq 0$. We will find in the next section that this quantity has a special name.

Definition 30.20 (Diagonal Matrix)**.** A *diagonal matrix* is a (square) matrix with the property that $\mathbf{D}_{ij} = 0$ for $i \neq j$ and $\mathbf{D}_{ii}$ may take any value in the field on which $\mathbf{D}$ is defined.

Remark 30.21. Thus, a diagonal matrix has (usually) non-zero entries only on its main diagonal. These matrices will play a critical roll in our analysis and are especially simple. They behave just like scalars (numbers) in their operations.

Definition 30.22 (Linear Independence)**.** A set of (column or row) vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent if the equation:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

just in case $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Put another way, if the only way to get the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ to add up to zero is to multiply them all by 0, then they are linearly independent.

Example 30.23. The vectors $\langle 1, 1\rangle$ and $\langle -1, 1\rangle$ are linearly independent. To see this let's solve for α_1 and α_2 so that:

$$\alpha_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This yields a system of equations:

$$\alpha_1 - \alpha_2 = 0$$
$$\alpha_1 + \alpha_2 = 0$$

We can see immediately that the only solution is $\alpha_1 = \alpha_2 = 0$. Thus the vectors are linearly independent. Vectors that are not linearly independent are called linearly dependent.

3. Determinants (Algebraically & Geometrically)

Definition 30.24 (Permutation)**.** A *permutation* on the set $\{1, \dots, n\}$ is a function $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. That is a permutation is just a way to mix the order of the sequence $1, 2, \dots, n$.

Example 30.25. There are 11 players on a cricket team. If we number the players $1, \dots, 11$, the order in which those 11 players bat is a permutation.

We might also consider a simple permutation on $\{1, 2, 3\}$ as:

$$\sigma = \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1. \end{matrix}$$

Thus $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$.

The identify permutation σ_{id} the permutation with $\sigma_{\text{id}}(k) = k$ for all $k \in \{1, \dots, n\}$.

Definition 30.26 (Transposition)**.** A *transposition* σ_{ij} on $\{1, \dots, n\}$ is a permutation that so that $\sigma(i) = j$ and $\sigma(j) = i$ and $\sigma(k) = k$ if $k \neq i$ and $k \neq j$. Thus it *transposes* i and j.

THEOREM 30.27. *Any permutation σ can be written as a composition of transpositions. Furthermore the parity (even or oddness) of the number of those compositions is unique and denoted* $\operatorname{sgn}(\sigma) \in \{-1, 1\}$ *with* $\operatorname{sgn}(\sigma) = 1$ *if and only if the number of compositions used is even.* □

Remark 30.28. The set of all permutations on the set $\{1, \dots, n\}$ is denoted S_n. The proof of the previous theorem is is covered in a course on abstract algebra or possibly a course on discrete mathematics. It is too far outside the scope of this course, but is relatively straightforward.

Definition 30.29 (Determinant)**.** Let $\mathbf{M} \in \mathbb{R}^{n\times n}$. The *determinant* of $\mathbf{M}$ is:

$$\det(\mathbf{A}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \mathbf{A}_{i\sigma(i)} \tag{30.6}$$

Here $\sigma \in S_n$ represents a permutation over the set $\{1, \dots, n\}$ and $\sigma(i)$ represents the value to which i is mapped under σ.

Example 30.30. Consider an arbitrary 2×2 matrix:

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

There are only two permutations in the set S_2: the identity permutation σ_{id} (which is even) and the transposition σ with $\sigma(1) = 2$ and $\sigma(2) = 1$, which is odd. Thus, we have:

$$\det(\mathbf{M}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathbf{M}_{11}\mathbf{M}_{22} - \mathbf{M}_{12}\mathbf{M}_{21} = ad - bc$$

This is the formula that one would expect from a course in matrices (like Math 220).

Remark 30.31. There is another way to think of a determinant, which does not require any algebra.

Definition 30.32 (Determinant)**.** Let $\mathbf{M} \in \mathbb{R}^{n\times n}$. The *determinant* of $\mathbf{M}$ is a mapping $\det : \mathbf{M} \mapsto V$, where V is the signed volume of the n-dimensional parallelpiped (parallelogram) formed by the rows of $\mathbf{M}$. Here *signed volume* just means that the volume may be negative depending on the location of the vectors in space.

Remark 30.33. If the rows of $\mathbf{M}$ do not form an n-dimensional shape, then the volume is 0.

Example 30.34. We can construct a formula for the determinant of a 2×2 matrix using only analytic geometry. Consider the matrix:

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The row vectors and the corresponding parallelogram formed by the rows are shown in Fig. 30.1. The area can be determined by finding the distance from the point $(a + c, b + d)$ that results from adding the vectors to the line extending from the vector $\langle c, d\rangle$. The formula for distance from the point (x_0, y_0) to the line determined by two points (x_1, y_1) and (x_2, y_2) is given by:

$$l = \frac{|(x_2 - x_1)(y_1 - y_0) - (x_1 - x_0)(y_2 - y_1)|}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

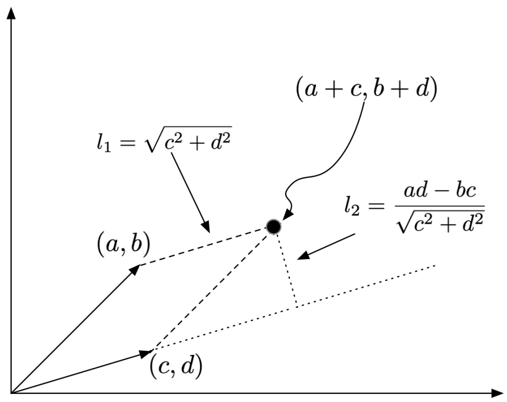


FIGURE 30.1. The determinant of the matrix is the area of the parallelogram formed by the row vectors, which can be determined from analytic geometry.

Using $(x_0, y_0) = (a+c, b+d)$, $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (c, d)$ yields the (signed) distance:

$$l_2 = \frac{ad - bc}{\sqrt{c^2 + d^2}}.$$

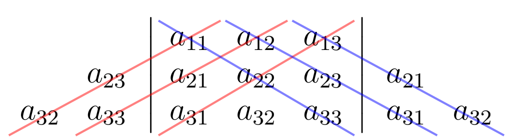
This is the height of the parallelogram. The width of the parallelogram is $l_1 = \sqrt{c^2 + d^2}$. Thus the area of the parallelogram is:

$$\det(\mathbf{M}) = A = l_1 l_2 = ad - bc.$$

Remark 30.35. There are formulas for finding determinants of larger matrices, but we will not consider matrices beyond 3×3. A course in Linear Algebra should discuss these methods. For a 3×3 matrix, the formula is:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32})\,.$$

This formula can be recovered using the following memory trick:



where you add the product of left-to-right diagonals and subtract the product of right-to-left diagonals.

Example 30.36. It's worth noting that a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent if and only if the determinant of the matrix with these vectors as columns (or rows) is non-zero if and only if the matrix is invertible. This makes much more sense from the geometric perspective. This can be illustrated with the vectors $\langle 1, 1\rangle$ and $\langle -1, 1\rangle$, which we already know are linearly independent. Note:

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1(1) - (-1)(1) = 2 \neq 0.$$

The fact that the matrix is invertible follows from the fact:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

which can be checked by multiplication.

4. Eigenvalues and Eigenvectors

Definition 30.37 (Eigenvalue and (Right) Eigenvector)**.** Let $\mathbf{M} \in \mathbb{R}^{n\times n}$. An eigenvalue, eigenvector pair $(\lambda, \mathbf{x})$ is a scalar and $n \times 1$ vector such that:

$$\mathbf{M}\mathbf{x} = \lambda\mathbf{x} \tag{30.7}$$

Remark 30.38. A *left eigenvector* is defined analogously with $\mathbf{x}^T\mathbf{M} = \lambda\mathbf{x}^T$, when $\mathbf{x}$ is considered a column vector. We will deal exclusively with right eigenvectors and hence when we say "eigenvector" we mean a right eigenvector.

Definition 30.39 (Characteristic Polynomial)**.** If $\mathbf{M} \in \mathbb{R}^{n\times n}$ then its *characteristic polynomial* is:

$$\det(\lambda\mathbf{I}_n - \mathbf{M}) \tag{30.8}$$

Remark 30.40. The following theorem is useful for computing eigenvalues of small matrices and defines the characteristic polynomial for a matrix.

THEOREM 30.41. *A value λ is an eigenvalue for $\mathbf{M} \in \mathbb{R}^{n\times n}$ if and only if it satisfies the characteristic equation:*

$$\det(\lambda\mathbf{I}_n - \mathbf{M}) = 0$$

Furthermore, $\mathbf{M}$ and $\mathbf{M}^T$ share eigenvalues. □

Example 30.42. Consider the matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The characteristic polynomial is computed as:

$$\det(\lambda\mathbf{I}_n - \mathbf{M}) = \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) - 0 = 0$$

Thus the characteristic polynomial for this matrix is:

$$\lambda^2 - 3\lambda + 2 \tag{30.9}$$

The roots of this polynomial are $\lambda_1 = 1$ and $\lambda_2 = 2$. Using these eigenvalues, we can compute eigenvectors:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{30.10}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{30.11}$$

and observe that:

$$\mathbf{M}\mathbf{x}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1\mathbf{x}_1 \tag{30.12}$$

and

$$\mathbf{M}\mathbf{x}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda_2 \mathbf{x}_2 \tag{30.13}$$

as required.

Example 30.43. Computing eigenvalues and eigenvectors by hand requires solving a system of linear equations with an infinite number of solutions. We illustrate the procedure on a 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

First we find the characteristic equation:

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - 1 = 0.$$

Solving for λ yields: $\lambda = 3$ or $\lambda = 1$. These are the two eigenvalues. We then seek a vector $\mathbf{x} = \langle x_1, x_2 \rangle$ so that:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \implies (\lambda \mathbf{I} - \mathbf{A})\,\mathbf{x} = \mathbf{0}.$$

This yields the equation:

$$\begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Substituting $\lambda = 3$ we have the system of equations:

$$\begin{aligned} x_1 - x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

These are the same equation, so we must make one variable a *free variable*. Say it's x_2 and we let $x_2 = t$ (where t stands for any real number). Then we see that $x_1 = x_2$ and so $x_1 = x_2 = t$. The result is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We have just deduced that the eigenvector for the eigenvalue $\lambda = 3$ is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector. You can scale this eigenvector anyway you want. So for example if you scale the eigenvector so it has unit norm (length) we would write:

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

This is because $\|\mathbf{x}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

If we do the same process for $\lambda = 1$ we obtain the system of equations:

$$\begin{aligned} -x_1 - x_2 &= 0 \\ -x_1 - x_2 &= 0 \end{aligned}$$

Again, letting $x_2 = t$ we see that $x_1 = -t$ and therefore:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Thus the eigenvector for the eigenvalue $\lambda = 1$ is:

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

which can again be scaled in anyway that is convenient.

Remark 30.44. It is important to remember that eigenvectors are unique *up to scale*. That is, if $\mathbf{M}$ is a square matrix and $(\lambda, \mathbf{x})$ is an eigenvalue eigenvector pair for $\mathbf{M}$, then so is $(\lambda, \alpha\mathbf{x})$ for $\alpha \neq 0$. This is because:

$$\mathbf{M}\mathbf{x} = \lambda\mathbf{x} \implies \mathbf{M}(\alpha\mathbf{x}) = \lambda(\alpha\mathbf{x}) \tag{30.14}$$

Remark 30.45. You can use your calculator to find the eigenvalues and eigenvectors of a matrix, as well as several software packages, like Matlab and Mathematica.

Definition 30.46 (Degenerate Eigenvalue & Multiplicity)**.** An eigenvalue is *degenerate* if it is a *multiple root* of the characteristic polynomial. The multiplicity of the root is the *algebraic multiplicity* of the eigenvalue. The *geometric multiplicity* of the eigenvalue is the number of eigenvectors it has.

Example 30.47. Consider the identify matrix $\mathbf{I}_2$. It has characteristic polynomial $(\lambda - 1)^2$, which has one multiple root 1. Thus $\lambda = 1$ is a degenerate eigenvalue for this matrix. However, this matrix does have two eigenvectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$. Thus the algebraic multiplicity of $\lambda = 1$ is 1, but the geometric multiplicity is 2.

LESSON 31

1. Linear Systems of ODE's

Remark 31.1. Recall Definition 1.26: A *system of ordinary differential equations* is a set of equations involving involving a set of unknown functions $y_1,\ldots,y_n$ and their derivatives each a function of one independent variable.

Remark 31.2. Before going on, it is worth noting that any order n differential equation can be transformed to an equivalent order $n-1$ system of differential equations. We illustrate this with a second order ODE:

$$\ddot{y} + a\dot{y} + b\dot{y} = f(t) \tag{31.1}$$

Define $v = \dot{y}$. Then the equation becomes $\dot{v} + av + by = f(t)$. We can then write a system of differential equations:

$$\begin{cases} \dot{v} = -av - by + f(t) \\ \dot{y} = v \end{cases} \tag{31.2}$$

By repeating this process, any order n differential equation can be written as a system of first order differential equations.

Remark 31.3 (Linear Homogeneous System)**.** Let $\mathbf{A}(t)$ be a matrix function of time so:

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

and let $\mathbf{b}(t)$ be an $n \times 1$ vector valued function. Then a *linear first order system* can be written as:

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t),$$

in the most general case. This is totally analogous to the first order ODE:

$$y' + p(t)y = q(t)$$

that we studied in the beginning of the semester. When $\mathbf{b}(t) = \mathbf{0}$, we have the *linear homogeneous system*:

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}.$$

Linear systems can be solved, but the solution is outside the scope of the course.

2. Linear Homogeneous Systems with Constant Coefficients

Definition 31.4 (Linear Homogeneous System with Constant Coefficients)**.** A linear homogeneous system with constant coefficients has form:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y},$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$. We assume that the rows (columns) of $\mathbf{A}$ are linearly independent.

Example 31.5. Consider the damped harmonic oscillator:

$$\ddot{x} + b\dot{x} + kx = 0.$$

Let $\dot{x} = v$. Then the corresponding first order system is:

$$\begin{aligned}\dot{x} &= v\\ \dot{v} &= -kx - bv.\end{aligned}$$

This can be written as:

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k & -b\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix}$$

Recall that that characteristic equation of the second order ODE is:

$$s^2 + bs + k = 0.$$

If we find the characteristic polynomial of the matrix defining the equivalent system of ODE's we see:

$$\begin{vmatrix}\lambda & -1\\ k & \lambda + b\end{vmatrix} = \lambda(\lambda + b) + k = \lambda^2 + b\lambda + k.$$

Thus the characteristic polynomial of the matrix is the same as the characteristic equation of the second order ODE. This is not an accident. We will see that the eigenvalues govern the solution of the system of ODE's.

Remark 31.6. Inspired by Lemma 16.14 we make the following definition.

Definition 31.7. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then:

$$e^{\mathbf{A}} = \exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^n.$$

Here $\mathbf{I}$ is an $n \times n$ identity matrix. Consequently:

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^n t^n. \tag{31.3}$$

Remark 31.8. We cannot formally prove the next lemma because it relies on certain facts from analysis about the uniform convergence of partial sums of Eq. (31.3), but nonetheless the approach is valid.

THEOREM 31.9. *The following derivative is valid:*

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}.$$

Consequently, $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$ *solves the IVP:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \mathbf{x}(0) = \mathbf{x}_0.$$

PROOF. Begin by differentiating $e^{\mathbf{A}t}$ term-by-term to see:

$$\frac{d}{dt}e^{\mathbf{A}t} = \frac{d}{dt}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^n t^n\right) = \sum_{n=0}^{\infty}\frac{d}{dt}\frac{1}{n!}\mathbf{A}^n t^n = \sum_{n=1}^{\infty}\frac{n}{n!}\mathbf{A}^n t^{n-1} =$$
$$\sum_{n=1}^{\infty}\mathbf{A}\left(\frac{1}{(n-1)!}\mathbf{A}^{n-1}t^{n-1}\right) = \mathbf{A}\left(\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\mathbf{A}^{n-1}t^{n-1}\right) =$$
$$\mathbf{A}\left(\sum_{n=0}^{\infty}\frac{1}{n!}A^n t^n\right) = \mathbf{A}e^{\mathbf{A}t}.$$

This establishes that $\exp(\mathbf{A}t)$ solves $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Notice that when $t = 0$, we have $\exp(\mathbf{0}) = \mathbf{I}$, the identity matrix. Consequently:

$$e^{\mathbf{A}t}\mathbf{x}_0\big|_{t=0} = \mathbf{I}\mathbf{x}_0 = \mathbf{x}_0.$$

Thus $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$ solves the IVP. □

Remark 31.10. We need to find a way to compute $e^{\mathbf{A}t}$ for a given matrix $\mathbf{A}$. We will discuss this in general, but then focus explicitly on the case when $\mathbf{A}$ is 2×2 because it is computationally simpler.

3. Diagonalization and Jordan's Decomposition Theorem

Definition 31.11 (Diagonalization)**.** Let $\mathbf{A}$ be an $n \times n$ matrix with entries from field $\mathbb{R}$. The matrix $\mathbf{A}$ can be diagonalized if there exists an $n \times n$ diagonal matrix $\mathbf{D}$ and another $n \times n$ matrix $\mathbf{P}$ so that:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} \tag{31.4}$$

In this case, $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is the *diagonalization* of $\mathbf{A}$.

Remark 31.12. Clearly if $\mathbf{A}$ is diagonalizable, then:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{31.5}$$

THEOREM 31.13. *A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is diagonalizable, if and only if the matrix has n (linearly independent) eigenvectors.* □

Remark 31.14. The proof of the theorem gives the recipe for diagonalizing a matrix.

PROOF. Suppose that $\mathbf{A}$ has a set of linearly independent eigenvectors $\mathbf{p}_1, \dots, \mathbf{p}_n$. Then for each $\mathbf{p}_i$, $(i = 1, \dots, n)$ there is an eigenvalue λ_i so that $\mathbf{A}\mathbf{p}_i = \lambda_i\mathbf{p}_i$. Let $\mathbf{P} \in \mathbb{C}^{n\times n}$ have columns $\mathbf{p}_1, \dots, \mathbf{p}_n$. Then we can see that:

$$\mathbf{A}\mathbf{P} = \left[\lambda_1\mathbf{p}_1|\cdots|\lambda_n\mathbf{p}_n\right] = \left[\mathbf{p}_1|\cdots|\mathbf{p}_n\right]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \cdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} = \mathbf{P}\mathbf{D},$$

where:

$$\mathbf{D} = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \cdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} \tag{31.6}$$

Since $\mathbf{p}_1, \dots, \mathbf{p}_n$ are linearly independent, it follows $\mathbf{P}$ is invertible and thus $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

Conversely, suppose that $\mathbf{A}$ is invertible and let $\mathbf{D}$ be as in Equation 31.6. Then:

$$\mathbf{AP} = \mathbf{DP}$$

and reversing the reasoning above, each column of $\mathbf{P}$ must be an eigenvector of $\mathbf{A}$ with corresponding eigenvalue on the diagonal of $\mathbf{D}$. □

Example 31.15. Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{31.7}$$

To diagonalize $\mathbf{A}$, we compute its eigenvalues and eigenvectors. The eigenvalues are computed as:

$$\begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

The two eigenvalues are: $\lambda = \pm i$. To compute the eigenvectors we must solve:

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{p} = \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

When $\lambda = i$ this yields the equations:

$$\begin{aligned} ip_1 + p_2 &= 0 \\ -p_1 + ip_2 &= 0 \end{aligned}$$

These equations are linearly dependent (multiply the first by i to recover the second). From the first equation we have $p_2 = -ip_1$. Let $p_1 = t$ and we see:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Thus the eigenvector corresponding to $\lambda = i$ is:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

On the other hand when $\lambda = -i$ we have the two equations

$$\begin{aligned} -ip_1 + p_2 &= 0 \\ -p_1 - ip_2 &= 0 \end{aligned}$$

Again, these equations are linearly dependent (multiply the first by $-i$ to recover the second). From the first equation we have $p_2 = ip_1$. Let $p_1 = t$ and we see:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Thus the eigenvector corresponding to $\lambda = i$ is:

$$\mathbf{p}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

We can now compute $\mathbf{P}$ and $\mathbf{D}$ as:

$$\mathbf{D} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

It is helpful to note that:

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

Arithmetic manipulation shows us that:

$$\mathbf{PD} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

Thus:

$$\mathbf{PDP}^{-1} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

as required.

LESSON 32

1. Computing Matrix Exponents

Lemma 32.1. *Let $\mathbf{D} \in \mathbb{R}^{n\times n}$ be a diagonal matrix with:*

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

then:

$$\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \tag{32.1}$$

Proof. Note that:

$$\mathbf{D}^2 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix}.$$

Now assume the Eq. (32.1) holds up to $k-1$. Then:

$$\mathbf{D}^k = \mathbf{D}^{k-1}\mathbf{D} = \begin{bmatrix} \lambda_1^{k-1} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{k-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^{k-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}.$$

The result follows by induction. □

Theorem 32.2. *Suppose $\mathbf{A}$ is diagonalizable and*

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

Then

$$\exp(\mathbf{A}t) = \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} \tag{32.2}$$

and if:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

then:

$$e^{\mathbf{D}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Proof. Note that:

$$\mathbf{A}^n t^n = \left(\mathbf{PD}t\mathbf{P}^{-1}\right)^n = (\mathbf{PD}t\mathbf{P}^{-1})(\mathbf{PD}t\mathbf{P}^{-1})\cdots(\mathbf{PD}t\mathbf{P}^{-1}) = \mathbf{P}(\mathbf{D}t)^n\mathbf{P}^{-1} \tag{32.3}$$

Using Definition 31.7 and the preceding equation we deduce that:

$$\sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{P}\left(\sum_{n=0}^{\infty} \frac{\mathbf{D}^n t^n}{n!}\right)\mathbf{P}^{-1}.$$

Now applying Lemma 32.1 and adding componentwise we see that there will be an infinite sum (that defines the exponential) along the diagonal elements. Thus:

$$e^{\mathbf{D}t} = \sum_{n=0}^{\infty} \frac{\mathbf{D}^n t^n}{n!} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}.$$

□

Theorem 32.3. *If* $\mathbf{A}$ *is diagonalizable and* $\mathbf{A} = \mathbf{PDP}^{-1}$*, then the solution to the linear homogeneous system:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \mathbf{x}(0) = \mathbf{x}_0$$

is:

$$\mathbf{x}(t) = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{x}_0. \tag{32.4}$$

Remark 32.4. We will return to the case when a nilpotent matrix arises in the Jordan decomposition when we need it.

2. Solving 2×2 Systems with Two Distinct Real Eigenvalues

Derivation 32.5. Practically speaking, if we apply Eq. (32.4) to a 2×2 matrix that the solution has form:

$$\mathbf{x}(t) = C_1\mathbf{p}_1 e^{\lambda_1 t} + C_2\mathbf{p}_2 e^{\lambda_2 t},$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the two distinct eigenvectors of the matrix $\mathbf{A}$. The values of C_1 can be determined from the initial conditions without inverting the matrix $\mathbf{P}$ (though you are certainly free to do so). In the case when $\lambda_1 \neq \lambda_2$ are real, this is the end of the story.

Example 32.6. Consider the linear system:

$$\begin{aligned}\dot{x} &= x + 2y\\ \dot{y} &= 2x + y.\end{aligned}$$

The matrix for this system is:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

To find the eigenvalues we solve:

$$\begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 4 = 0.$$

We see that $\lambda = 3$ and $\lambda = -1$ are eigenvalues. To find the eigenvectors we take each eigenvalue in turn.

Case $\lambda = 3$: We solve:

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{p} = \begin{bmatrix} 3-1 & -2 \\ -2 & 3-1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two equations are:

$$\begin{aligned}2p_1 - 2p_2 &= 0\\ -2p_1 + 2p_2 &= 0\end{aligned}$$

So $p_1 = p_2$ and we can set $p_1 = t$. Then:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the first eigenvector is:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Case $\lambda = -1$: We solve:

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{p} = \begin{bmatrix} -1-1 & -2 \\ -2 & -1-1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two equations are:

$$\begin{aligned}-2p_1 - 2p_2 &= 0\\ -2p_1 - 2p_2 &= 0\end{aligned}$$

So $p_2 = -p_1$ and we can set $p_1 = t$. Then:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the second eigenvector is:

$$\mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Solution: The solution is then:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

If we are given $x(0) = 2$ and $y(0) = 0$, then we would solve:

$$C_1 + C_2 = 2$$
$$C_1 - C_2 = 0$$

and we conclude that $C_1 = C_2 = 1$. You can verify this by computing:

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and using Eq. (32.4).

1. Solving 2×2 Systems with Two Complex Eigenvalues

Remark 33.1. We can do a bit more analysis when the eigenvalues are complex. This can help avoid doing a lot of complex arithmetic and also illustrates the nature of the solutions.

Lemma 33.2. *If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a real matrix with complex eigenvalue λ and corresponding complex eigenvector $\mathbf{p}$, then $\bar{\lambda}$ and $\bar{\mathbf{p}}$ (complex conjugates) are also an eigenvalue/eigenvector pair.*

PROOF. Notice that:

$$\mathbf{A}\mathbf{p} = \lambda\mathbf{p},$$

which implies that:

$$\bar{\mathbf{A}}\bar{\mathbf{p}} = \bar{\lambda}\bar{\mathbf{p}} = \mathbf{A}\bar{\mathbf{p}} = \bar{\lambda}\bar{\mathbf{p}},$$

because $\mathbf{A}$ is real and the conjugation operation distributes over multiplication (which you can check). Thus $\bar{\lambda}$ is an eigenvalue and $\bar{\mathbf{p}}$ is its eigenvector. □

Corollary 33.3. *If $\mathbf{A} \in \mathbb{R}^{2\times 2}$ has a complex eigenvalue λ with (complex) eigenvector $\mathbf{p}$, then its other eigenvector is $\bar{\lambda}$ with corresponding eigenvector $\bar{\mathbf{p}}$.* □

Derivation 33.4. Suppose:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and this has two complex eigenvalues $\lambda + \mu i$ and $\lambda - \mu i$. The two corresponding eigenvalues must be conjugate as we have:

$$\mathbf{p}_1 = \begin{bmatrix} r_1 + s_1 i \\ r_2 + s_2 i \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} r_1 - s_1 i \\ r_2 - s_2 i \end{bmatrix}.$$

Then:

$$\mathbf{P} = \begin{bmatrix} r_1 + s_1 i & r_1 - s_1 i \\ r_2 + s_2 i & r_2 - s_2 i \end{bmatrix} \qquad \mathbf{P}^{-1} = \frac{1}{2i}\frac{1}{r_2 s_1 - r_1 s_2}\begin{bmatrix} r_2 - s_2 i & -r_1 + s_1 i \\ -r_2 - s_2 i & r_1 + s_1 i \end{bmatrix}.$$

Notice that $1/i = -i$ because $i(-i) = 1$. So we can write this as:

$$\mathbf{P}^{-1} = \frac{1}{2(r_2 s_1 - r_1 s_2)}\begin{bmatrix} -r_2 i - s_2 & r_1 i + s_1 \\ r_2 i - s_2 & -r_1 i + s_1 \end{bmatrix}$$

We know that:

$$e^{\mathbf{D}t} = \begin{bmatrix} e^{(\lambda+\mu i)t} & 0 \\ 0 & e^{(\lambda-\mu i)t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} \cos(\mu t) + i\sin(\mu t) & 0 \\ 0 & \cos(\mu t) - i\sin(\mu t) \end{bmatrix}$$

We can now compute the solution to:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

We have:

$$\begin{bmatrix}x\\ y\end{bmatrix} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \frac{e^{\lambda t}}{2D}\begin{bmatrix}2B\sin(\mu t) + 2D\cos(\mu t) & -2l_1\sin(\mu t)\\ 2l_2\sin(\mu t) & 2D\cos(\mu t) - 2B\sin(\mu t)\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix},$$

where:

$$\begin{aligned}
l_1 &= r_1^2 + s_1^2\\
l_2 &= r_2^2 + s_2^2\\
D &= r_2 s_1 - r_1 s_2\\
B &= r_1 r_2 + s_1 s_2.
\end{aligned}$$

Simplifying we conclude:

$$\begin{bmatrix}x\\ y\end{bmatrix} = \frac{e^{\lambda t}}{D}\begin{bmatrix}\sin(\mu t)\,(x_0 B - y_0 l_1) + x_0 D\cos(\mu t)\\ \sin(\mu t)\,(x_0 l_2 - y_0 B) + y_0 D\cos(\mu t)\end{bmatrix}.$$

Notice this is a real-valued solution. We now have the following theorem.

THEOREM 33.5. *Consider the system of ordinary differential equations:*

$$\begin{aligned}
\dot{x} &= ax + by\\
\dot{y} &= cx + dy
\end{aligned}$$

and suppose that the system has two complex eigenvalues $\lambda \pm \mu i$ with corresponding eigenvectors:

$$\mathbf{p}_{1,2} = \begin{bmatrix}r_1 \pm s_1 i\\ r_2 \pm s_2 i\end{bmatrix}$$

Then the general solution is given by:

$$\begin{aligned}
x(t) &= \frac{e^{\lambda t}}{D}\left\{C_1\left[D\cos(\mu t) + B\sin(\mu t)\right] - C_2 l_1 \sin(\mu t)\right\}\\
y(t) &= \frac{e^{\lambda t}}{D}\left\{C_2\left[D\cos(\mu t) - B\sin(\mu t)\right] + C_1 l_2 \sin(\mu t)\right\},
\end{aligned}$$

where:

$$\begin{aligned}
l_1 &= r_1^2 + s_1^2\\
l_2 &= r_2^2 + s_2^2\\
D &= r_2 s_1 - r_1 s_2\\
B &= r_1 r_2 + s_1 s_2.
\end{aligned}$$

□

Remark 33.6. Other texts will give a slightly simpler formulation in terms of the real and imaginary parts of the solution derived from one of the eigenvectors. This is equivalent but then requires an analysis with the Wronskian:

$$W(\mathbf{X}) = \det\left([\mathbf{x}_1, \ldots, \mathbf{x}_n]\right),$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are distinct vector valued solutions. Since we have solved the ODE explicitly (and for all cases) using the matrix exponential, this analysis is not needed. For the simpler formula, see Theorem 36.4.

Example 33.7. Reconsider the harmonic oscillator as a system:

$$\begin{aligned}\dot{x} &= v\\ \dot{v} &= -\omega^2 x\end{aligned}$$

This has matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1\\ -\omega^2 & 0\end{bmatrix}$$

Of course this leads to the characteristic equation $\lambda^2 = -\omega^2$ and two complex eigenvalues $\lambda = \pm i\omega$. We can solve for one of the eigenvectors by solving:

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{p} = \mathbf{0},$$

or

$$\begin{bmatrix} i\omega & -1\\ \omega^2 & i\omega\end{bmatrix}\begin{bmatrix} p_1\\ p_2\end{bmatrix} = \begin{bmatrix} 0\\ 0\end{bmatrix}$$

We have the two equations:

$$\begin{aligned} i\omega p_1 - p_2 &= 0\\ \omega^2 p_1 + i\omega p_2 &= 0\end{aligned}$$

The first equation can be converted into the second equation by multiplying by $-i\omega$; they are linearly dependent. Then from the first equation we see:

$$p_2 = i\omega p_1.$$

Let $p_1 = t$ we have the solution:

$$\begin{bmatrix} p_1\\ p_2\end{bmatrix} = t\begin{bmatrix} 1\\ i\omega\end{bmatrix}.$$

Then we know the two eigenvectors must be:

$$\mathbf{p}_1 = \begin{bmatrix} 1\\ i\omega\end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1\\ -i\omega\end{bmatrix}$$

So from our theorem we have $r_1 = 1$ and $s_1 = 0$ and $r_2 = 0$ and $s_2 = \omega$. We can compute:

$$\begin{aligned} l_1 &= 1\\ l_2 &= \omega^2\\ D &= -\omega\\ B &= 0.\end{aligned}$$

The eigenvalue is purely imaginary so we have no exponential term. Assume $x(0) = x_0$ and $v(0) = v_0$. Then the solution to the ODE is:

$$\begin{aligned} x(t) &= \frac{1}{\omega}\left[x_0\omega\cos(\omega t) + v_0\sin(\omega t)\right]\\ v(t) &= \frac{1}{\omega}\left[v_0\omega\cos(\omega t) - x_0\omega^2\sin(\omega t)\right].\end{aligned}$$

Note we have simplified some signs in the above solution.

LESSON 34

1. Solving 2×2 Systems with only One Eigenvector and a Repeated Eigenvalue

Remark 34.1. We must go back to dealing with the case when a matrix is not diagonalizable. This will happen when an eigenvalue has arithmetic multiplicity 2 and geometric multiplicity 1. To do this, we introduce a generalization of diagonalizing a matrix.

Definition 34.2 (Nilpotent Matrix)**.** A matrix $\mathbf{N}$ is *nilpotent* if there is some integer $k > 0$ so that $\mathbf{N}^k = \mathbf{0}$

Example 34.3. The matrix:

$$\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is nilpotent. To see this compute:

$$\mathbf{N}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Remark 34.4. We generalize the notion of *diagonalization* in a concept called the *Jordan Normal Form*. The proof of the Jordan Normal Form theorem is outside the scope of the class, but it can be summarized in the following theorem. See [**Lan12**] for details.

Theorem 34.5. *Let* $\mathbf{A}$ *be a square matrix with complex entries (i.e.,* $\mathbf{A} \in \mathbb{C}^{n\times n}$*). Then there exists matrices* $\mathbf{P}$*,* $\mathbf{D}$ *and* $\mathbf{N}$ *so that: (1)* $\mathbf{D}$ *is a diagonal matrix with the eigenvalues of* $\mathbf{A}$ *appearing on the diagonal. (2)* $\mathbf{N}$ *is a nilpotent matrix and (3)* $\mathbf{P}$ *is a matrix whose columns are composed of pseudo-eigenvectors and (4):*

$$\mathbf{A} = \mathbf{P}(\mathbf{D}+\mathbf{N})\mathbf{P}^{-1}, \tag{34.1}$$

When $\mathbf{A}$ *is diagonalizable, then* $\mathbf{N} = \mathbf{0}$ *and* $\mathbf{P}$ *is a matrix whose columns are composed of eigenvectors.*

Derivation 34.6. When $\mathbf{A} \in \mathbb{R}^{n\times n}$ is not diagonalizable, we use Theorem 34.5 to compute $e^{\mathbf{A}t}$. Recall:

$$\mathbf{A} = \mathbf{P}\left(\mathbf{D}+\mathbf{N}\right)\mathbf{P}^{-1},$$

where $\mathbf{N}$ is a nilpotent matrix and $\mathbf{D}$ is a matrix with eigenvalues on the diagonal. Applying the same reasoning as in the proof of Theorem 32.2 we see that:

$$e^{\mathbf{A}t} = \mathbf{P}e^{(\mathbf{D}+\mathbf{N})t}\mathbf{P}^{-1}.$$

Now:

$$e^{(\mathbf{D}+\mathbf{N})t} = e^{\mathbf{D}t}e^{\mathbf{N}t}.$$

We already know what $e^{\mathbf{D}t}$ from Theorem 32.2. To compute $e^{\mathbf{N}t}$ we use Definition 31.7:

$$\exp(\mathbf{N}t) = \sum_{m=0}^{\infty} \frac{\mathbf{N}^m t^m}{m!}.$$

Since $\mathbf{N}$ is nilpotent, we know there is a k so that $\mathbf{N}^k = \mathbf{0}$. Thus if $m \geq k$, $\mathbf{N}^m = \mathbf{0}$ and the infinite sum becomes a finite sum:

$$\exp(\mathbf{N}t) = \sum_{m=0}^{k} \frac{\mathbf{N}^m t^m}{m!}.$$

We conclude that the solution to:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \mathbf{x}(0) = \mathbf{x}_0$$

is:

$$\mathbf{x}(t) = \mathbf{P} \exp\left(\mathbf{D}t\right) \left(\sum_{m=0}^{k} \frac{\mathbf{N}^m t^m}{m!} \right) \mathbf{P}^{-1} \cdot \mathbf{x}_0,$$

when $\mathbf{A}$ is not diagonalizable. In particular, this means that we expect to see polynomials in t appearing multiplied by exponentials in the solution.

Derivation 34.7. In the case of a 2×2 homogeneous linear system with an eigenvalue with algebraic multiplicity 2 (repeated root) and geometric multiplicity 1 (one eigenvector) we *always* have:

$$\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This fact comes out of the proof of Jordan Canonical Form, so it's outside the scope of the notes. Since we know $\mathbf{N}^2 = \mathbf{0}$ from Example 34.3, we conclude that:

$$e^{\mathbf{N}t} = \mathbf{I} + t\mathbf{N} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

If λ is the repeated eigenvalue, we know that:

$$e^{\mathbf{D}t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix},$$

so:

$$e^{\mathbf{D}t} e^{\mathbf{N}t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t}. \end{bmatrix}$$

It is now simply a matter of finding $\mathbf{P}$. If $\mathbf{p}_1$ is the eigenvector corresponding to λ, that will be the first column of $\mathbf{P}$. The question of how to find the second column is (fortunately or unfortunately) also answered in the proof of the Jordan theorem. For us it suffices to solve:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{p}_2 = \mathbf{p}_1.$$

Notice, this is different in sign from how we've been finding eigenvectors. This sign difference doesn't matter for eigenvectors (when the right hand side is zero) but it does matter here.

There will be an infinite number of solutions to this problem and it's best to take the simplest possible one. Just as in Derivation 32.5, we can multiply $\mathbf{P}e^{\mathbf{D}t}e^{\mathbf{N}t}$ and use constants to see that the general solution can be written as:

$$\mathbf{x} = C_1 \mathbf{p}_1 e^{\lambda t} + C_2 \left(\mathbf{p}_1 t e^{\lambda t} + \mathbf{p}_2 e^{\lambda t} \right),$$

for arbitrary constants C_1 and C_2.

Example 34.8. Consider the following linear system of differential equations[1]:

$$\dot{x} = 7x + y$$
$$\dot{y} = -4x + 3y$$

The matrix is:

$$\mathbf{A} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$$

We compute:

$$\begin{vmatrix} \lambda - 7 & -1 \\ 4 & \lambda - 3 \end{vmatrix} = (\lambda - 7)(\lambda - 3) + 4 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0.$$

The eigenvalue in question is $\lambda = 5$ and it has algebraic multiplicity 2. Let us find its eigenvetor(s). We compute:

$$(\lambda I - \mathbf{A})\,\mathbf{p} = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We conclude that:

$$-2p_1 - p_2 = 0$$

This implies $p_2 = -2p_1$ and if $p_1 = t$ yields the solution:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

so the eigenvector is:

$$\mathbf{p} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We can find no other eigenvector for this eigenvalue. So we must find a pseudo-eigenvector by solving:

$$(\mathbf{A} - \lambda I)\,\mathbf{p}_2 = \mathbf{p}_1.$$

This leads to the equation:

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We now have the two equations:

$$2p_1 + p_2 = 1$$
$$-4p_1 - 2p_2 = -2.$$

These equations are linearly dependent so letting $p_2 = t$ and using the first equation yields:

$$2p_1 = -t + 1 \implies p_1 = \frac{1}{2} - \frac{t}{2}.$$

[1]This example is taken from http://tutorial.math.lamar.edu/Classes/DE/RepeatedEigenvalues.aspx, where it is presented differently.

We can write:

$$\mathbf{p}_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Now we pick the easy solution and set $t = 0$. We obtain the matrix:

$$\mathbf{P} = \begin{bmatrix} 1 & \frac{1}{2} \\ -2 & 0 \end{bmatrix},$$

and our solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{5t} + C_2 \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} te^{5t} \right).$$

Now suppose $x(0) = 2$ and $y(0) = -5$. Then:

$$C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

We have the equation system:

$$\begin{aligned} C_1 + \tfrac{1}{2}C_2 &= 2 \\ -2C_1 &= -5 \end{aligned}$$

Thus $C_1 = \frac{5}{2}$ and $C_2 = -1$. Multiplying out solution is:

$$\begin{aligned} x(t) &= 2e^{5t} - te^{5t} \\ y(t) &= -5e^{5t} + 2te^{5t}. \end{aligned}$$

Remark 34.9. It is worth nothing that

$$\mathbf{P}^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 1 \end{bmatrix},$$

and as we expect:

$$\begin{bmatrix} 1 & \frac{1}{2} \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}.$$

However there is an additional step to choosing the correct psuedo-eigenvector representation to make this happen. That is, when we happily set $t = 0$ in finding $\mathbf{p}_2$ in the previous example, that was designed to make sure the Jordan form resulted. The arbitrary constants C_1 and C_2 always clean up any mess made by selecting a different pseudo-eigenvector in the differential equations. However, if you check for another problem you may find the Jordan theorem fails for your choice of pseudo-eigenvector. This does not mean you're wrong, it means you need to choose a different version of that pseudo-eigenvector.

LESSON 35

1. Fixed Points

Remark 35.1. We return (momentarily) to a more general autonomous system of differential equations:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \dots, x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, \dots, x_n),\end{aligned}$$

which can be written more compactly as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{35.1}$$

where $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function of a vector of inputs.

Definition 35.2 (Fixed Points)**.** Consider the system of differential equations given by Expression 35.1. A vector $\mathbf{x}^* \in \mathbb{R}^n$ is a *fixed* or *equilibrium* point of Expression 35.1 if $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$.

Remark 35.3. The notions of fixed point stability that we studied in Lesson 6, Section 2 carry over with absolute values replaced by vector norms, since both the dependent and independent variables are now presented in vector form. For example, a fixed point $\mathbf{x}^*$ is stable if for all $\epsilon > 0$ there is a $\delta > 0$ so that when $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$, then $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for all $t \geq 0$. Here $\mathbf{x}(t)$ is a solution to the system of differential equations.

Proposition 35.4. *If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a matrix with linearly independent rows (or columns), then the linear homogeneous (autonomous) system:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

has one fixed point $\mathbf{x}^* = \mathbf{0}$.

Remark 35.5. The stability of this single fixed point is completely characterized by the eigenvalues of the system and there are only (effectively) four qualitative behaviors that can be exhibited by solutions to a linear homogeneous system. However, most text break these up into six classifications. These are best understood geometrically.

2. Phase Portraits of Linear Systems

Definition 35.6 (Phase Portrait)**.** Consider System 35.1:

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, \dots, x_n)\\ \vdots \\ f_n(x_1, \dots, x_n)\end{bmatrix}$$

and suppose we have a general solution $x_1(t; x_1^0), \dots x_n(t; x_n^0)$, parameterized by the initial values $x_1^0, \dots, x_n^0$. (That is, suppose we have a specific orbit.) Then a *phase portrait* is a geometric representation of the behavior of the differential equation system obtained by creating multiple parametric curves $(x_1(t; x_1^0), \dots x_n(t; x_n^0)$ for various starting values $x_1^0, \dots, x_n^0$; i.e., We are visualizing multiple orbits of the system.

Remark 35.7. It is worth noting that some people consider the phase portrait to be the vector field obtained from the differential system. So at any point $x_1, \dots, x_n \in \mathbb{R}^n$, the vector:

$$\begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix}$$

is computed and displayed with some appropriate scaling (so that enormous vectors do not distort the visual effect.) In either case, it is worth letting a computer draw the phase portrait for you.

Example 35.8 (Phase Portraits of Linear Systems - Sink and Source)**.** Consider the linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

A *sink* occurs when $\mathbf{A}$ has two negative real eigenvalues, while a *source* occurs when $\mathbf{A}$ has two real positive eigenvalues. These are illustrated in Fig. 35.1. In a sink the fixed point

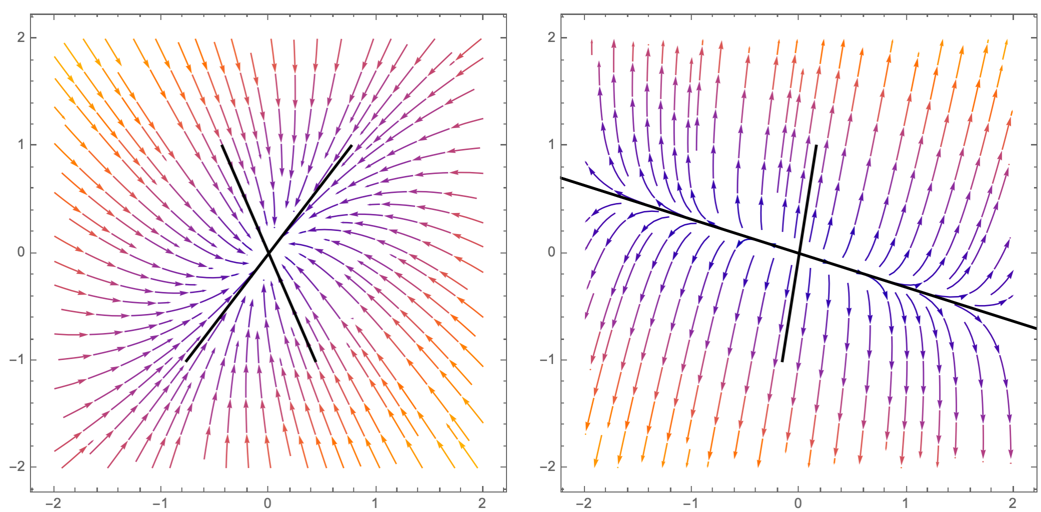


FIGURE 35.1. (Left) A sink occurs where there are two real negative eigenvalues. The eigenvectors (shown in black) for the system form a non-perpendicular axis that the flow follows near the fixed point. (Right) A source occurs where there are two real positive eigenvalues. The eigenvectors (shown in black) for the system form a non-orthogonal axis that the flow follows near the fixed point.

$\mathbf{x} = \mathbf{0}$ is asymptotically stable while in a source it is asymptotically unstable. Flow follows the system's eigenvectors, which act like a non-orthogonal (non-perpendicular) axis.

Example 35.9 (Phase Portraits of Linear Systems - Spiral Sink and Spiral Source)**.** A *spiral sink* occurs when $\mathbf{A}$ has two complex eigenvalues with negative real part, while a *spiral source* occurs when $\mathbf{A}$ has two complex eigenvalues with positive real part. These are illustrated in Fig. 35.2. In a spiral sink the fixed point $\mathbf{x} = \mathbf{0}$ is asymptotically stable

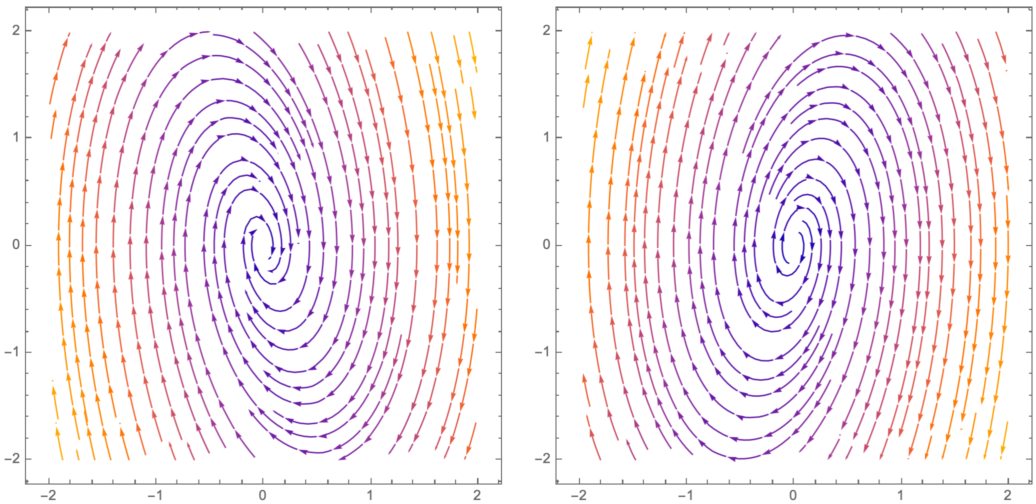


FIGURE 35.2. (Left) A spiral sink occurs where there are two complex eigenvalues with negative real part. (Right) A spiral source occurs where there are two complex eigenvalues both with positive real part.

while in a spiral source it is asymptotically unstable. Notice we have not shown the system eigenvectors because they do not play as large a role in the flow direction. This mirrors our analysis where we used the (complex) eigenvectors but they do not appear explicitly in the solution as they do in the other cases.

Example 35.10 (Phase Portraits of Linear Systems - Saddle and Center)**.** A *saddle* occurs when $\mathbf{A}$ has two real eigenvalues that are opposite in sign (i.e., one positive and one negative eigenvalue) while a *center* occurs when the two eigenvalues are both pure imaginary (have zero real part). This is illustrated in Fig. 35.3. In a saddle, the fixed point $\mathbf{x} = \mathbf{0}$ is unstable, but not asymptotically unstable. In a center, the fixed point $\mathbf{x} = \mathbf{0}$ is stable but asymptotically stable.

Remark 35.11. Despite the different names, we can see there are really four behaviors:

(1) The fixed point $\mathbf{x} = \mathbf{0}$ can be asymptotically stable.
(2) The fixed point $\mathbf{x} = \mathbf{0}$ can be asymptotically unstable.
(3) The fixed point $\mathbf{x} = \mathbf{0}$ can be a saddle (unstable).
(4) The fixed point $\mathbf{x} = \mathbf{0}$ can be a center (stable).

These are the four possible qualitative behaviors for the fixed point in a linear system. Some texts call a fixed point that is a center an *elliptic* fixed point and all other fixed points *hyperbolic* fixed points. This extends to nonlinear systems as well.

Remark 35.12. We can summarize the possible behaviors with a theorem whose proof (in two dimensions) follows from our constructed solutions. However, using Eq. (32.4) and Derivation 34.6 the proof can be extended to all first order linear systems.

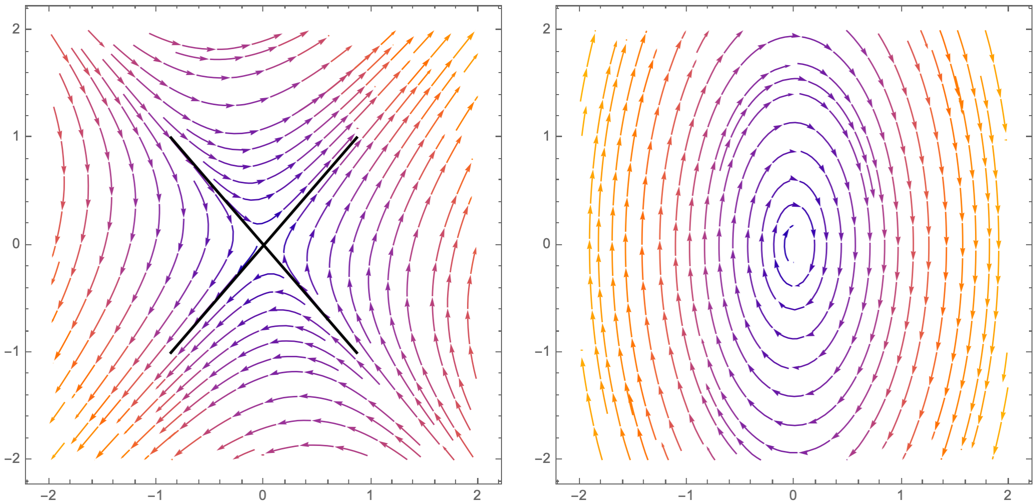


FIGURE 35.3. (Left) A saddle occurs where there are two real eigenvalues with opposite sign. The eigenvectors (shown in black) for the system form a non-perpendicular axis that the flow follows near the fixed point. (Right) A center occurs when both eigenvalues are pure imaginary numbers.

THEOREM 35.13. *Consider the linear homogeneous sytem:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

The following hold:

(1) *The fixed point* $\mathbf{x} = \mathbf{0}$ *is asymptotically stable if all the eigenvalues of* $\mathbf{A}$ *have negative real part.*
(2) *The fixed point* $\mathbf{x} = \mathbf{0}$ *is asymptotically unstable if all the eigenvalues of* $\mathbf{A}$ *have positive real part.*
(3) *The fixed point* $\mathbf{x} = \mathbf{0}$ *is an (unstable) saddle if the real parts of the eigenvalues of* $\mathbf{A}$ *have mixed sign. (In the two-dimensional case, these will be two real eigenvalues of opposite sign.)*
(4) *The fixed point* $\mathbf{x} = \mathbf{0}$ *is a stable center if all the eigenvalues of* $\mathbf{A}$ *are imaginary or zero.*

□

Example 35.14. It is worth asking what the phase portrait looks like when there is only one real eigenvalue. This is illustrated for the system:

$$\dot{x} = 7x + y$$
$$\dot{y} = -4x + 3y,$$

which we studied in Example 34.8. This system has a single positive eigenvalue $\lambda = 5$ and so we expect $\mathbf{x} = \mathbf{0}$ to be unstable, which it is. However only one direction of flow can be determined using the single true eigenvector. Flow in other directions varies.

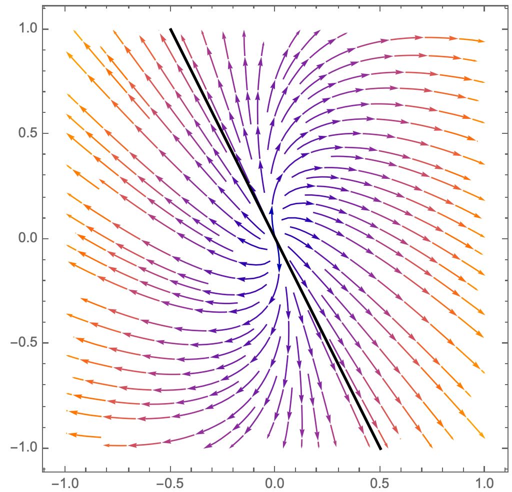


FIGURE 35.4. The repeated root case follows the rules of stability based on the sign of the eigenvalue. However, only one direction of flow can be determined by the single true eigenvector (shown in black).

LESSON 36

1. Three Examples of 2×2 Systems of ODE's

Example 36.1. Suppose we wish to solve:

$$\dot{x} = 3x - 6y$$
$$\dot{y} = 5x - 8y,$$

which we can rewrite as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

Step 1 - Compute Eigenvalues: The first step to solving this problem is finding the eigenvalues of the matrix. We have:

$$\begin{vmatrix} \lambda - 3 & 6 \\ -5 & \lambda + 8 \end{vmatrix} = (\lambda - 3)(\lambda + 8) + 30 = \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0.$$

The eigenvalues are $\lambda = -2$ and $\lambda = -3$.

Step 2 - Compute Eigenvectors: We first compute the eigenvector for $\lambda = -3$. We have the system of equations:

$$\begin{bmatrix} -6 & 6 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then the eigenvector equation is:

$$-6p_1 + 6p_2 = 0,$$

this implies $p_1 = p_2$. Setting $p_1 = t$ we have:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $t = 1$. Then the eigenvector corresponding to $\lambda = -3$ is $\mathbf{p}_1 = \langle 1, 1 \rangle$. To compute the eigenvector corresponding to $\lambda = -2$, we have:

$$\begin{bmatrix} -5 & 6 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then the eigenvector equation is:

$$-5p_1 + 6p_2 = 0,$$

this implies $p_1 = \frac{6}{5}p_2$. Setting $p_2 = t$ we have:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} \frac{6}{5} \\ 1 \end{bmatrix}.$$

Setting $t = 5$ we have $\mathbf{p}_2 = \langle 6, 5 \rangle$ as the eigenvector corresponding to $\lambda = -2$.

Step 3 - Write General Solution: The general solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 6 \\ 5 \end{bmatrix} e^{-2t}.$$

Step 4 - Find Specific Solutions: Substituting in the initial conditions yields

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

Therefore we have:

$$\begin{aligned} 7 &= C_1 + 6C_2 \\ 6 &= C_1 + 5C_2 \end{aligned}$$

We conclude $C_1 = 1$ and $C_2 = 1$. Therefore the specific solution is:

$$\begin{aligned} x(t) &= e^{-3t} + 6e^{-2t} \\ y(t) &= e^{-3t} + 5e^{-2t}. \end{aligned}$$

Example 36.2. Suppose we wish to solve:

$$\begin{aligned} \dot{x} &= -2x - 4y \\ \dot{y} &= x - 6y, \end{aligned}$$

which we can rewrite as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Step 1 - Compute Eigenvalues: The first step to solving this problem is finding the eigenvalues of the matrix. We have:

$$\begin{vmatrix} \lambda + 2 & 4 \\ -1 & \lambda + 6 \end{vmatrix} = (\lambda + 2)(\lambda + 6) + 4 = \lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0.$$

The single eigenvalue is $\lambda = -4$.

Step 2 - Compute Eigenvector(s): We compute the eigenvector for $\lambda = -4$. We have the system of equations:

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then the eigenvector equation is:

$$-2p_1 + 4p_2 = 0,$$

this implies $p_1 = 2p_2$. Setting $p_2 = t$ we have:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Let $t = 1$. Then the eigenvector is $\mathbf{p}_1 = \langle 2, 1 \rangle$. We now need to compute a linearly independent pseudo-eigenvector. Letting:

$$\mathbf{A} = \begin{bmatrix} -2 & -4 \\ 1 & -6 \end{bmatrix}$$

To this we must solve $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{p}_2 = \mathbf{p}_1$. This leads to the system:

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Using the second row, the eigenvector equation is:

$$p_1 - 2p_2 = 1$$

this implies $p_1 = 1 + 2p_2$. Setting $p_2 = t$ we have:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Setting $t = 0$ we have $\mathbf{p}_2 = \langle 1, 0 \rangle$ as the pseudo-eigenvector.
Step 3 - Write General Solution: The general solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-4t} + C_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-4t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-4t} \right)$$

Step 4 - Find Specific Solutions: Substituting in the initial conditions yields:

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore we have:

$$\begin{aligned} 4 &= 2C_1 + C_2 \\ 3 &= C_1 \end{aligned}$$

We conclude $C_1 = 3$ and $C_2 = -2$. Therefore the specific solution is:

$$x(t) = 4e^{-4t} - 4te^{-4t} \qquad y(t) = 3e^{-4t} - 2te^{-4t}.$$

Remark 36.3. Before we proceed to the final example, we will state a theorem that will be proved as an exercise.

Theorem 36.4. *Consider the system of ordinary differential equations given by:*

$$\mathbf{x} = \mathbf{A}\mathbf{x},$$

and assume $\mathbf{A} \in \mathbb{R}^{2\times 2}$ *has complex conjugate eigenvalues* $s = \lambda \pm \mu i$ *with corresponding complex conjugate eigenvectors:*

$$\mathbf{p}_1 = \begin{bmatrix} r_1 + s_1 i \\ r_2 + s_2 i \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} r_1 - s_1 i \\ r_2 - s_2 i \end{bmatrix}.$$

Then the general solution to the system of ordinary differential equations is:

$$\mathbf{x} = C_1 e^{\lambda t} \left(\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \cos(\mu t) - \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \sin(\mu t) \right) + C_2 e^{\lambda t} \left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \cos(\mu t) + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \sin(\mu t) \right).$$

□

Example 36.5. Suppose we wish to solve:

$$\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= x - y, \end{aligned}$$

which we can rewrite as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step 1 - Compute Eigenvalues: The first step to solving this problem is finding the eigenvalues of the matrix. We have:

$$\begin{vmatrix} s+1 & 1 \\ -1 & s+1 \end{vmatrix} = (s+1)^2 + 1 = 0.$$

The eigenvalues is $s = -1 \pm i$. We see $\lambda = -1$ and $\mu = 1$.

Step 2 - Compute Eigenvector(s): We compute the eigenvector for $s = -1+i$. We have the system of equations:

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then the eigenvector equation is:

$$ip_1 + p_2 = 0,$$

this implies $p_2 = -ip_1$. Setting $p_1 = t$ we have:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Let $t = 1$. Then the eigenvector is $\mathbf{p}_1 = \langle 1, -i \rangle$. With this information we have:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1+0i \\ 0-1i \end{bmatrix},$$

so $r_1 = 1$, $r_2 = 0$, $s_1 = 0$, $s_2 = -1$.

Step 3 - Write General Solution: The general solution is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{-t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin(t) \right) + C_2 e^{-t} \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t) \right).$$

Simplifying yields:

$$x(t) = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$$
$$y(t) = C_1 e^{-t} \sin(t) - C_2 e^{-t} \cos(t)$$

Step 4 - Find Specific Solutions: Substituting in the initial conditions yields:

$$x(0) = 1 = C_1$$
$$y(0) = 1 = -C_2,$$

therefore $C_1 = 1$ and $C_2 = -1$. Therefore the specific solution is:

$$x(t) = e^{-t} \cos(t) - e^{-t} \sin(t)$$
$$y(t) = e^{-t} \sin(t) + e^{-t} \cos(t).$$

Module 8

Introduction to Linear Partial Differential Equations

LESSON 37

1. Two Lemma's on Boundary Value Problems

Remark 37.1. Our objective in this section is to state two lemma's that will be needed for solving the heat and wave equation in one dimension.

Remark 37.2. We state our differential equations in terms of spatial variable x with unknown function $w(x)$. However it would be just as easy to use $w(t)$ and this would not affect the results. Additionally, we use the notation $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ to denote the set of positive integers and $\mathbb{Z}_0^+ = \{0, 1, 2, \dots\}$ to denote the set of non-negative integers.

Lemma 37.3. *Consider the ordinary second order differential equation with boundary conditions:*

$$\begin{cases} w'' + \lambda w = 0 \\ w(0) = w(L) = 0, \end{cases} \tag{37.1}$$

where $L > 0$. This equation has non-trivial real solutions:

$$w_n(x) = b_n \sin\left(\frac{n\pi}{L} x\right).$$

with $b_n \in \mathbb{R}$ only if $\lambda > 0$ and $\lambda = (n\pi/L)^2$ for $n \in \mathbb{Z}^+$.

PROOF. We divide the proof into three cases.

Case 1: $\lambda = 0$:
In this case, the differential equation becomes:

$$w'' = 0.$$

This can be solved by direct integration (twice) to obtain:

$$w(x) = C_1 x + C_2.$$

If $w(0) = 0$, then $C_2 = 0$. If $w(L) = 0$, then $C_1 = 0$. Thus $w(x) = 0$ is the trivial solution.

Case 2: $\lambda < 0$:
The characteristic polynomial of the ordinary differential equation is:

$$s^2 + \lambda = 0. \tag{37.2}$$

Therefore, the solutions all have form:

$$w(x) = C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x}. \tag{37.3}$$

For simplicity, assume $\lambda = -\sigma^2$. Then Eq. (37.3) becomes:

$$w(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x}. \tag{37.4}$$

Using the boundary condition $w(0) = 0$ and substituting we see that:

$$(37.5) \quad C_1 + C_2 = 0.$$

Therefore we can rewrite Eq. (37.4) as:

$$(37.6) \quad w(x) = \frac{C}{2}\left(e^{\sigma x} - e^{-\sigma x}\right) = C\sinh(\sigma x),$$

for some constant C. Here, sinh is the hyperbolic sine function defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

This function has a single root at $x = 0$; i.e., $\sinh(0) = 0$. Therefore, since $L > 0$ it follows that:

$$C\sinh(L) = 0 \implies C = 0.$$

Thus, $w(x) = 0$ is the trivial solution.

Case 2: $\lambda > 0$:
Reasoning as before, Eq. (37.3) still holds, but now assume that $\lambda = \sigma^2$. Then the solution becomes:

$$(37.7) \quad w(x) = C_1\sin(\sigma x) + C_2\cos(\sigma x).$$

With the requirement that $w(0) = 0$, we see that $C_2 = 0$ since $\cos(0) = 1$. Thus, the solution has form:

$$(37.8) \quad w(x) = C_1\sin(\sigma x).$$

If $w(L) = 0$, we must have $\sin(\sigma L) = 0$. This can only happen if:

$$\sigma L = n\pi,$$

for $n \in \mathbb{Z}^+$ because $\sin(n\pi) = 0$. (We can ignore the negative integers, since we can use C_1 to adjust the sign.) Therefore, this equation has non-trivial solution only when:

$$\sigma = \frac{n\pi}{L},$$

which implies:

$$(37.9) \quad \lambda = \left(\frac{n\pi}{L}\right)^2.$$

The result is a non-trivial family of solutions:

$$w_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right),$$

where b_n are arbitrary constants to be determined later. This completes the proof. □

Remark 37.4. Thus we see that with the boundary conditions $w(0) = w(L)$ and linear operator:

$$L = \frac{d^2}{dx^2}$$

the eigenvalues and eigenfunctions of L are:

$$\lambda = -\left(\frac{n\pi}{L}\right)^2$$

with eigenfunctions:

$$w_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right).$$

This can be verified by showing that:

$$\frac{d^2w}{dx^2} = \lambda w.$$

In understanding this, it's important to not get confused by the signs. Here, λ is *negative* to make Definition 3.25 true (where the operator $\frac{d^2}{dx^2}$. Whereas in Eq. (37.1) from Lemma 37.3 the differential equation under consideration is:

$$w'' = -\lambda w.$$

That is, the negative is already taken into consideration. To eliminate the sign confusion, many texts simply refer to this as the eignvalue problem.

Lemma 37.5. *Consider the ordinary second order differential equation with boundary conditions:*

$$\begin{cases} w'' + \lambda w = 0 \\ w'(0) = w'(L) = 0, \end{cases} \tag{37.10}$$

where $L > 0$. This equation has non-trivial real solutions

$$w_n(x) = a_n \cos\left(\frac{n\pi}{L}x\right) \tag{37.11}$$

with $a_n \in \mathbb{R}$ only if $\lambda > 0$ and $\lambda = (n\pi/L)^2$ for $n \in \mathbb{Z}^+$ or $\lambda = 0$ and $w_0(x) = a_0$.

Remark 37.6. The proof of this is almost a direct copy of the previous proof. Given that, we will skip the case when $\lambda < 0$ and leave it as an exercise.

Proof. Assume $\lambda > 0$. It follows from the proof of Lemma 37.3 that Eq. (37.7) sill holds and:

$$w(x) = C_1 \sin(\sigma x) + C_2 \cos(\sigma x),$$

where $\lambda = \sigma^2$. Computation shows:

$$w'(t) = \sigma C_1 \cos(\sigma x) - \sigma C_2 \sin(\sigma x). \tag{37.12}$$

If $w'(0) = 0$, then $C_1 = 0$. If $w'(L) = 0$, then by a similar argument as the one in the proof of Lemma 37.3 we see that if:

$$\sigma = \frac{n\pi}{L},$$

with $n \in \mathbb{Z}^+$, then:

$$-\sigma C_2 \sin(\sigma x) = 0.$$

Thus when $\lambda > 0$, the non-trivial family of solutions to the boundary value problem is:

$$w_n(x) = a_n \cos\left(\frac{n\pi}{L}x\right).$$

Now assume $\lambda = 0$. Then if $w(x) = a_0$ for some $a_0 \in \mathbb{R}$, we see that $w'(x) = 0$ for all x, satisfying the boundary conditions. Substitution shows this solution satisfies the ODE. □

Remark 37.7. From the above proof, we can see that the eigenvalues for the boundary values problem given in Eq. (37.10) are:

$$\lambda = -\left(\frac{n\pi}{L}\right)^2$$

while the eigenfunctions are given in Eq. (37.11) or the eigenvalue $\lambda = 0$ with the eigenfunctions a_0 for any value a_0.

LESSON 38

1. Solving the Homogeneous Heat Equation with Separation of Variables

Remark 38.1. For complete details on PDE's, see (e.g.) [**Asm16**, **Hab03**, **Olv14**].

Definition 38.2 (Initial and Boundary Conditions of the Heat Equation)**.** Recall the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

which models the evolution of temperature on an infinitely thin rod. An *initial condition* $u(x,0) = f(x)$ specifies the heat distribution at the beginning of time, while *Dirichlet boundary conditions* $u(0,t) = u(L,t) = T$ specify the temperature at the two end-points. Alternatively *Neumann boundary conditions* $u_x(0,L) = u_x(L,t) = v$ specify the heat flow at the boundary.

Definition 38.3 (Homogeneous Boundary Conditions)**.** The boundary conditions $u(0,t) = u(L,t) = 0$ or $u_x(0,t) = u_x(L,t)$ are *homogeneous* and correspond to having our thin rod with ends in blocks of ice (Dirichlet boundary conditions) or being insulated (Neumann boundary conditions).

Remark 38.4. It is possible to model more complex boundary conditions, but we will not consider that in this course.

Remark 38.5. The remainder of this lesson will be devoted to solving the heat equation using the separation of variables techniques. We stress, this approach works for the homogeneous heat equation with homogeneous boundary conditions (Dirichlet or Neumann) and periodic boundary conditions or Robin boundary conditions (which will not discuss in this class).

Remark 38.6. For the remainder of this lesson, we will omit the initial condition and return to it when we discuss Fourier series in the next chapter.

Derivation 38.7. We consider the one-dimensional heat equation with homogeneous Dirichlet boundary condition:

$$\begin{aligned} &u_t = ku_{xx} \\ &u(0,t) = u(L,t) = 0 \end{aligned}$$

Assume (for the sake of argument) that $u(x,t) = v(t)w(x)$, where $v(t)$ and $w(x)$ are two C^2 univariate functions of time and space respectively. If $u(x,t)$ satisfies the heat equation, then:

$$u_t = v'(t)w(x) = kv(t)w''(x) = ku_{xx}.$$

From this we deduce that:

$$\frac{1}{k}\frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)}.$$

The left-hand-side of this equation is strictly a function of t while the right-hand-side is strictly a function of x. Therefore, if this were to hold for all x and t, it follows that

$$\frac{1}{k}\frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)} = -\lambda,$$

where $\lambda \in \mathbb{R}$ is a constant and we use $-\lambda$ to make it consistent with our lemmas. We now have two ordinary differential equations:

Time ODE:

$$v' = -k\lambda v$$

Space ODE:

$$w'' = -\lambda w$$
$$w(0) = w(L) = 0.$$

Notice we have imposed the boundary conditions on the space ODE because *for all* t, $u(0,t) = u(L,t) = 0$, so unless $v(t) = 0$ (i.e., we have a trivial solution), we'll require $w(0) = w(L) = 0$.

From Lemma 37.3, we know there is a family of solutions to the space ODE given by:

$$w_n(x) = a_n \sin\left(\frac{n\pi}{L}x\right).$$

with:

$$\lambda = \left(\frac{n\pi}{L}\right)^2.$$

We can now solve the time ODE. Recall from Remark 3.6, the solution to the time ODE is:

$$v(t) = A\exp\left(-k\lambda t\right). \tag{38.1}$$

Substituting in the value of λ (as a function of n) we have:

$$v_n(t) = A_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right],$$

where A_n is an arbitrary constant that we are going to absorb into the b_n. Note that the the operator $L = \partial_t - k\partial_{xx}$ is linear. Therefore we can add the individual solutions $u_n(x,t) = v_n(t)w_n(x)$ to obtain a generic solution to the heat equation with homogeneous Dirichlet boundary values:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\sin\left(\frac{n\pi}{L}x\right). \tag{38.2}$$

Thus we have proved a proposition.

Proposition 38.8. *Given coefficients b_n ($n \in \mathbb{Z}^+$), if the series:*

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\sin\left(\frac{n\pi}{L}x\right). \tag{38.3}$$

may be differentiated term-by-term in both x and t, then $u(x,t)$ solves the one-dimensional heat equation with homogeneous Dirichlet boundary conditions. $\square$

Remark 38.9. We are not going to worry a great deal about whether or not term-by-term differentiation is allowed in the solution $u(x,t)$ defined in Eq. (38.3). Suffice it to say, it is allowed for all the choices of b_n we require.

Remark 38.10. It is worth noting now that we have not determined *how* to identify the coefficients b_n. This will be a major topic of discussion when we introduce initial conditions and discuss Fourier series and Fourier decomposition in the next chapter.

Proposition 38.11. *Given coefficients a_n for $n = 0, 1, \dots$ if the series*

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\cos\left(\frac{n\pi}{L}x\right). \tag{38.4}$$

may be differentiated term-by-term in both x and t, then $u(x,t)$ solves the heat equation with Neumann boundary conditions:

$$\begin{aligned}
&u_t = ku_{xx}\\
&u_x(0,t) = u_x(L,t) = 0
\end{aligned}$$

Remark 38.12. It is possible to model the heat equation on a circle in which case we use periodic boundary conditions $u(-L,t) = u(L,t)$. The result is a solution with both sines and cosines.

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\sin\left(\frac{n\pi}{L}x\right).$$

The derivation is straightforward but relies on another boundary value problem lemma.

LESSON 39

1. Definitions and Preliminaries

Remark 39.1. In this lesson, we will follow historical precedent and define Fourier series on the interval $[-L, L]$. In doing so, we will construct formulae for determining the coefficients a_0, a_n and b_n from the previous chapter. However, this should create some tension in your mind because we spent the last chapter solving problems on $[0, L]$. We can easily translate solutions back and forth or we can find formulae for our coefficients on the interval $[0, L]$. We will illustrate both approaches.

Definition 39.2 (Even and Odd Functions)**.** A function $f(x)$ is *even* if $f(-x) = f(x)$. It is *odd* if $f(-x) = -f(x)$.

Example 39.3. The classic examples of even and odd functions are $f(x) = x^2$ and $f(x) = x$ respectively. Recall from trigonometry that $\cos(x)$ is an even function while $\sin(x)$ is an odd function.

2. Fourier Series

Definition 39.4 (Fourier Series)**.** A *Fourier Series* defined on the interval $[-L, L]$ is an infinite series of the form:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \tag{39.1}$$

Remark 39.5. Much like a Taylor series for a function $f(x)$, the idea is to construct a Fourier series for a function so that:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

This is called the *Fourier decomposition* of the function. We will see, however, that this is not always possible to enforce equality in the strictest sense. Therefore, following [**Olv14**, **Hab03**], we we will write:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

to indicate that equality may not hold for every value of x.

Lemma 39.6. *Suppose $m, n \in \mathbb{Z}_+$. Then:*

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\, dx = \begin{cases} 0 & \textit{if } m \neq n \\ L & \textit{otherwise.} \end{cases} \tag{39.2}$$

Proof. Assume $n \neq m$. Apply the trigonometric identity:

$$\sin(\theta)\sin(\varphi) = \frac{1}{2}\left[\cos(\theta-\varphi) - \cos(\theta+\varphi)\right].$$

The integral in Eq. (39.2) becomes:

$$\int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{n-m}{L}\pi x\right) - \cos\left(\frac{n+m}{L}\pi x\right)\right]\, dx = \\ \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{n-m}{L}\pi x\right) - \frac{L}{(n+m)\pi}\sin\left(\frac{n+m}{L}\pi x\right)\right]\Bigg|_{-L}^{L}$$

Using $\sin(-\theta) = -\sin(\theta)$, we can simplify the result as:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\, dx = \frac{L}{2\pi}\left\{2\sin[(n-m)\pi] - 2\sin[(n+m)\pi]\right\}.$$

If $n \neq m$, then $n-m$ and $n+m$ are non-zero integers and thus $\sin[(n-m)\pi] = \sin[(n+m)\pi] = 0$.

Now suppose $n = m$. Apply the trigonometric identity:

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

Then:

$$\int_{-L}^{L} \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right) = \frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Bigg|_{-L}^{L} = L.$$

This completes the proof. □

Lemma 39.7. *Suppose $m, n \in \mathbb{Z}_0^+$. Then:*

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } n = m \neq 0 \\ 2L & \text{if } n = m = 0. \end{cases} \tag{39.3}$$

Lemma 39.8. *Suppose $m, n \in \mathbb{Z}_0^+$. Then:*

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\, dx = 0 \tag{39.4}$$

Proof. Apply the identity:

$$\sin(\theta)\cos(\varphi) = \frac{1}{2}\left[\sin(\theta+\varphi) + \sin(\theta-\varphi)\right].$$

The integral becomes:

$$\frac{1}{2}\int_{-L}^{L} \sin\left[\frac{(n+m)\pi x}{L}\right] + \sin\left[\frac{(n-m)\pi x}{L}\right]\, dx = \\ \frac{1}{2}\left\{-\frac{L}{(n+m)\pi}\cos\left[\frac{(n+m)\pi x}{L}\right] - \frac{L}{(n-m)\pi}\cos\left[\frac{(n-m)\pi x}{L}\right]\right\}\Bigg|_{-L}^{L}.$$

Cosine is an even function so $\cos(-x) = \cos(x)$. Therefore, evaluating at $x = L$ and $x = -L$ we see:

$$\cos\left[(n+m)\pi\right] - \cos\left[-(n+m)\pi\right] = 0 = \cos\left[(n-m)\pi\right] - \cos\left[-(n-m)\pi\right].$$

Thus the integral is zero as required. $\square$

Proposition 39.9. *Suppose that* $f : [-L, L] \to \mathbb{R}$ *and*

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

Then:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \tag{39.5}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \tag{39.6}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \tag{39.7}$$

assuming these integrals exist and we may integrate the Fourier series term-by-term.

Proof. If $m = 0$, then $1 = \cos(m\pi x/L)$. It follows that:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \left[a_0 \cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\right] dx$$

Passing the integrals through the sums and applying Lemmas 39.7 and 39.8 implies:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} a_0\,dx = 2La_0.$$

Therefore:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

For $m \neq 0$, we compute:

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\,dx = \int_{-L}^{L} \left[a_0 \cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\right] dx$$

Now suppose that $m = n$. Applying Lemmas 39.7 and 39.8 again implies:

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \int_{-L}^{L} a_n \cos^2\left(\frac{n\pi x}{L}\right)\,dx = a_nL,$$

because (again) $a_0 = a_0 \cos(m\pi x/L)$ for $m = 0$. Therefore:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx.$$

The equality

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx,$$

holds by a similar argument but by multiplying by $\sin(n\pi x/L)$. $\square$

Remark 39.10. In order for a function to have a Fourier series, these integrals must exist for all coefficients. In particular, this means that $f(x)$ must at least be integrable on $[-L, L]$ (or an appropriate interval in question). Thus, (for example) this eliminates function with asymptotes.

LESSON 40

1. Completing the Solution to the Heat Equation on $[0, L]$

Remark 40.1. Having established some basic theory on Fourier series, we can now return to the problem of solving the heat equation with an initial condition $u(x,0) = f(x)$.

Lemma 40.2. *If $f : [0, L] \to \mathbb{R}$ is a function and*

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

then:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

□

Remark 40.3. The representation of $f(x)$ on $[0, L]$ using only sines is called a *Fourier sine series.*

Remark 40.4. It is straightforward to prove Lemma 40.2 by first showing that

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n. \end{cases}$$

Then we argue as we did in Proposition 39.9.

Derivation 40.5. Consider the one-dimensional heat equation with homogeneous Dirichlet boundary conditions and an initial condition:

$$\begin{aligned} &u_t = k u_{xx} \\ &u(0,t) = u(L,t) = 0 \\ &u(x,0) = f(x). \end{aligned}$$

Recall the solution was given in Eq. (38.2):

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right] \sin\left(\frac{n\pi}{L}x\right).$$

except for values of b_n. Note we require:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

Therefore, we are simply constructing a Fourier decomposition of $f(x)$ using sine functions. We can apply Lemma 40.2 to obtain the values for b_n and we have solved the partial differential equation completely.

Example 40.6. Consider the heat equation with $k = 1$ and $L = 1$ and homogeneous Dirichlet boundary conditions. Suppose:

$$u(x,0) = \begin{cases} 1 & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Then:

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\int_{\frac{1}{4}}^{\frac{3}{4}} \sin(n\pi x)\,dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_{\frac{1}{4}}^{\frac{3}{4}} =$$
$$\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right)$$

We can use a sum-to-product identity

$$\cos(\theta) - \cos(\varphi) = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

to compute:

$$\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right) = \frac{-4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(-\frac{n\pi}{4}\right) = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{4}\right),$$

because $\sin(-x) = \sin(x)$. It is perfectly acceptable to stop here (or even the step before) and write:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{4}\right)\sin(n\pi x). \tag{40.1}$$

The approximation for $f(x)$ using 20 terms is shown in Fig. 40.1. Suppose, however, we wish

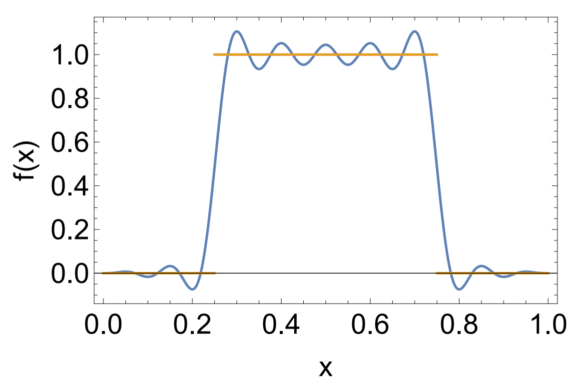


FIGURE 40.1. An approximation of the step function describing the initial heat distribution.

to get rid of those trigonometric terms. We can do it, but it's not obvious from the outside. First, note $n = 2m$ (i.e., even) then $b_n = 0$ because:

$$\sin\left(\frac{n\pi}{2}\right) = \sin(m\pi) = 0.$$

Thus, we will be left with only odd terms. When n is odd, then $n = 2m-1$ (for $m = 1, 2, \dots$) we have:

$$\sin\left(\frac{(2m-1)\pi}{2}\right)\sin\left(\frac{(2m-1)\pi}{4}\right) = (-1)^{f(m)}\frac{\sqrt{2}}{2}.$$

The exact form of $f(m)$ is unclear. Computing a few terms we see a pattern: It turns out

m	1	2	3	4	5	6	7	8
$\sin\left(\frac{(2m-1)\pi}{4}\right)$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

this patten can be described by the function:

$$\sin\left(\frac{(2m-1)\pi}{2}\right)\sin\left(\frac{(2m-1)\pi}{4}\right) = (-1)^{\frac{m(m-1)}{2}}\frac{\sqrt{2}}{2}$$

Therefore, we can write:

$$f(x) \sim \sum_{m=1}^{\infty}(-1)^{\frac{m(m-1)}{2}}\frac{2\sqrt{2}}{(2m-1)\pi}\sin\left((2m-1)\pi x\right).$$

We conclude that:

$$u(x,t) = \sum_{m=1}^{\infty}(-1)^{\frac{m(m-1)}{2}}\frac{2\sqrt{2}}{(2m-1)\pi}\sin\left[(2m-1)\pi x\right]\exp\left[-k\left((2m-1)\pi\right)^2 t\right].$$

Assuming $k = 1/10$, an approximation using 100 terms for various times is shown in Fig. 40.2.

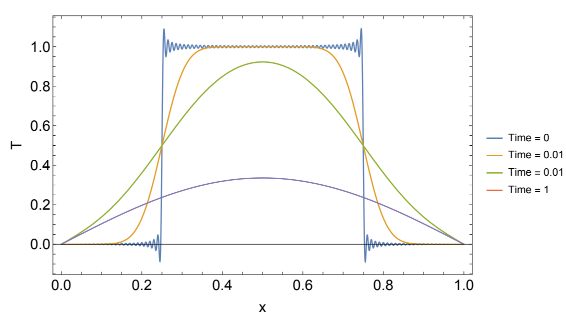


FIGURE 40.2. An approximation of the solution to the heat equation when the initial distribution has a jump discontinuity.

Remark 40.7. It is worth noting that the jump discontinuity could be at the boundary even given the Dirichlet boundary conditions. That is, we could use the initial condition:

$$u(x,0) = \begin{cases} 10 & \text{if } 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

and use the boundary condition $u(0,t) = u(0,L) = 0$.

Remark 40.8. We note also this is an example of a solution that is not really differentiable everywhere at all times and yet we are using it as a solution to a PDE.

Lemma 40.9. *If $f : [0, L] \to \mathbb{R}$ is a function and*

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

then:

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

□

Remark 40.10. The representation of $f(x)$ on $[0, L]$ using only cosines is called a *Fourier cosine series.*

Derivation 40.11. Consider the one-dimensional heat equation with homogeneous Neumann boundary conditions and an initial condition:

$$u_t = ku_{xx}$$
$$u_x(0, t) = u_x(L, t) = 0$$
$$u(x, 0) = f(x).$$

Recall the solution was given in Eq. (38.4):

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\cos\left(\frac{n\pi}{L}x\right).$$

except for values of a_0 and a_n. Note we require:

$$u(x, 0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right).$$

Therefore, we are simply constructing a Fourier decomposition of $f(x)$ using cosine functions. We can apply Lemma 40.9 to obtain the values for a_0 and a_n and we have solved the partial differential equation completely.

Example 40.12. Consider the same problem data as in Example 40.6 except replace the homogeneous Dirichlet boundary conditions with homogeneous Neumann boundary conditions. Then:

$$a_0 = \int_{\frac{1}{4}}^{\frac{3}{4}} dx = \frac{1}{2}.$$

Likewise:

$$a_n = 2\int_{\frac{1}{4}}^{\frac{3}{4}} \cos(n\pi x)\,dx = \frac{2}{n\pi}\left[\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right].$$

Simplifying further is left to the reader (if so desired). The resulting solution is:

$$u(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right]\exp\left[-k\,(n\pi)^2\,t\right]\cos\left(n\pi x\right)$$

We can illustrate the resulting solution with the density plot in Fig. 40.3.

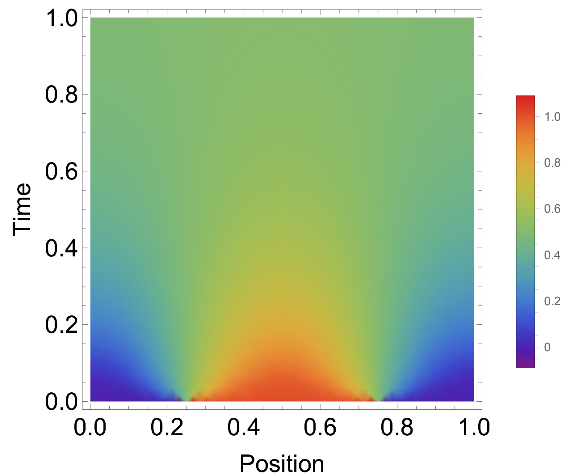


FIGURE 40.3. An illustration of the heat equation with Neumann boundary conditions and a discontinuous initial condition.

LESSON 41

1. Separation of Variables Solution for the Wave Equation

Proposition 41.1 (String with Fixed Ends)**.** *Consider the one-dimensional wave equation with Dirichlet boundary conditions:*

$$\begin{aligned}
&u_{tt} = c^2 u_{xx}\\
&u(0,t) = u(L,t) = 0\\
&u(x,0) = f(x)\\
&u_t(x,0) = g(x).
\end{aligned}$$

Then the solution to this PDE is given by:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right), \tag{41.1}$$

where a_n is given by Lemma 40.2 as:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

and b_n is given by:

$$b_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

□

Example 41.2. Consider the wave equation with Dirichlet boundary conditions on the interval $[0,\pi]$ and let:

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \leq \pi. \end{cases}$$

Suppose $g(x) = 0$; i.e., if we are modeling the string, it is initially at rest. Then: $b_n = 0$ and we must only compute a_n.

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \\
&\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin(nx)\,dx = \begin{cases} \frac{4}{n^2\pi}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, we can write:

$$u(x,t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4}{(2n-1)^2 \pi} \cos\left[(2n-1)ct\right] \sin\left[(2n-1)x\right]$$

Time snapshots of the resulting solution with $c = 1$ are shown in Fig. 41.1.

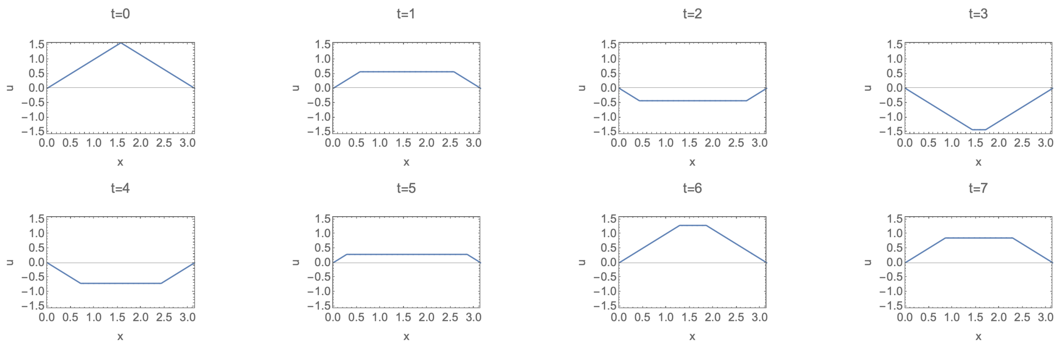


FIGURE 41.1. A solution to the wave equation is illustrated assuming $c = 1$. This models a string plucked into a triangle and then released.

Derivation 41.3. Consider the wave equation on the interval $[0, L]$ with Neumann boundary conditions. This derivation is a bit harder than the case with Dirichlet boundary conditions, which is left as an exercise. The wave equation is second order in t as well as x, so our initial conditions must specify both $u(x, 0)$ and $u_t(x, 0)$. Thus our fully specified problem is:

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \\ &u_x(0,t) = u_x(L,t) = 0 \\ &u(x,0) = f(x) \\ &u_t(x,0) = g(x). \end{aligned}$$

We approach this with separation of variables again. Let:

$$u(x,t) = v(t)w(x).$$

Then the boundary conditions become:

$$\begin{aligned} &w'(0) = 0 \\ &w'(L) = 0. \end{aligned}$$

Differentiating (as before) we have:

$$v''(t)w(x) = c^2 v(t) w''(x) \implies \frac{1}{c^2}\frac{v''(t)}{v(t)} = \frac{w''(x)}{w(x)}.$$

As before the left-hand-side is only a function of t while the right-hand-side is only a function of x. This yields the equations:

$$\begin{aligned} \frac{1}{c^2}\frac{v''}{v} &= -\lambda \\ \frac{w''}{w} &= -\lambda, \end{aligned}$$

where λ is a constant and the sign is chosen to be consistent with Lemmas 37.3 and 37.5. The second equation provides the boundary value problem:

$$w'' + \lambda w = 0$$
$$w'(0) = 0$$
$$w'(L) = 0.$$

We know from Lemma 37.3 this has solutions:

$$w_n(x) = a_n \cos(n\pi x/L) \tag{41.2}$$

with eigenvalues $\lambda = (n\pi/L)^2$ or $w_0(x) = a_0$ for $\lambda = 0$.

Case I: ($\lambda > 0$) Consider the case when $\lambda > 0$. We now turn our attention to the problem for $v(t)$, which on simplification becomes:

$$v'' + \left(\frac{c^2 n^2 \pi^2}{L^2}\right) v = 0.$$

If we assume $c > 0$, then from the proof of Lemma 37.3 (or Lemma 37.5) we know the solutions have form:

$$v_n(t) = A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right). \tag{41.3}$$

Case II: ($\lambda = 0$) On the other hand, when $\lambda = 0$, the problem for $v(t)$ is:

$$v'' = 0$$

From the proof Lemma 37.3, we know the solutions have form:

$$v_0(t) = C_0 + C_1 t. \tag{41.4}$$

Combining Solutions: Combining Eq. (41.2) with Eq. (41.3) we have:

$$u_n(x,t) = v_n(t)w_n(t) = a_n \cos\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

When $\lambda = 0$, we combine Eq. (41.4) with the solution $w_0(x) = a_0$ to obtain:

$$u_0(x,t) = a_0 + c_1 t.$$

Summing up all solutions we see:

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty} a_n \cos\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right). \tag{41.5}$$

We need only find the values of a_0, c_1, a_n and b_n. Evaluating Eq. (41.5) at $t = 0$ we have:

$$u(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

because $\cos(0) = 1$ and $\sin(0) = 0$. Therefore, we must find a Fourier cosine series expansion of $f(x)$ on the interval $[0, L]$. The coefficients a_0 and a_n are given in Lemma 40.9.

Turning now to the second initial condition, we differentiate $u(x,t)$ term-by-term with respect to t to obtain:

$$u_t(x,t) = b_0 + \sum_{n=1}^{\infty} -a_n \frac{cn\pi}{L} \sin\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + b_n \frac{cn\pi}{L} \cos\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

Now evaluating at $t = 0$:

$$u_t(x, 0) = g(x) = b_0 + \sum_{n=1}^{\infty} b_n \frac{cn\pi}{L} \cos\left(\frac{n\pi x}{L}\right).$$

We seek a Fourier cosine series for the function $h(x) = Lg(x)/cn\pi$. Therefore we have:

$$b_n = \frac{2}{L} \int_0^L h(x) \cos\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L} \int_0^L \frac{L}{cn\pi} g(x) \cos\left(\frac{n\pi x}{L}\right)\, dx.$$

Thus we conclude:

$$b_n = \frac{2}{cn\pi} \int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right)\, dx.$$

The coefficient b_0 is also given in Lemma 40.9 and is:

$$b_0 = \frac{1}{L} \int_0^L g(x)\, dx.$$

Remark 41.4. Solutions to the wave equation with Neumann boundary conditions have the interesting property that they can become unbounded. This is a result of the $b_0 t$ term that appears in the solution. This is clearly a non-physical solution.

LESSON 42

1. Interesting Fourier Series Examples

Example 42.1. We will find a Fourier series for the function:

$$f(x) = \frac{x^2}{4}$$

on the interval $[-\pi, \pi]$. Using this interval will remove the π terms in the functions and get us an interesting corollary. First we compute b_n. We can use an integral table or a computer algebra system (like Maple or Mathematica) to compute:

$$\int \frac{x^2}{4}\sin(nx)\,dx = \frac{2x\sin(nx)}{4n^2} - \frac{(n^2x^2-2)\cos(nx)}{4n^3}.$$

Notice the function:

$$g(x) = \frac{(n^2x^2-2)\cos(nx)}{4n^3}$$

is *even* because $\cos(x)$ and $ax^2 - b$ are both even. Therefore evaluating $g(x)$ at $x = \pi$ and $x = -\pi$ yields the same values. At the same time, $\sin(n\pi) = 0$ and therefore we can conclude that:

$$\int_{-\pi}^{\pi} \frac{x^2}{4}\sin(nx)\,dx = \frac{2x\sin(nx)}{4n^2} - \frac{(n^2x^2-2)\cos(nx)}{4n^3} = 0.$$

If you've already used a computer, you could have jumped right to this step. Therefore, $b_n = 0$ for all n. This is not surprising. We know $f(x)$ is an even function so we would not expect it to be the result of adding up odd functions like $\sin(nx)$.

Next we compute a_n for $n \geq 1$. We have:

$$\int \frac{x^2}{4}\cos(nx)\,dx = \frac{(n^2x^2-2)\sin(nx)}{4n^3} + \frac{2x\cos(nx)}{4n^2}.$$

Evaluating at $x = -\pi$ and $x = \pi$ and subtracting yields:

$$\int_{-\pi}^{\pi} \frac{x^2}{4}\cos(nx)\,dx = \frac{4\pi\cos(\pi n)}{4n^2},$$

because $\sin(n\pi) = 0$ and $x\cos(x)$ is odd. This can be simplified by noting that $\cos(\pi n) = (-1)^n$. Therefore we conclude:

$$a_n = \frac{1}{\pi}\left(\frac{(-1)^n\pi}{n^2}\right) = \frac{(-1)^n}{n^2}.$$

for $n \geq 1$. Finally:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{x^2}{4}\,dx = \frac{\pi^2}{12}.$$

We conclude that:

$$\frac{x^2}{4} \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

This is illustrated in Fig. 42.1.

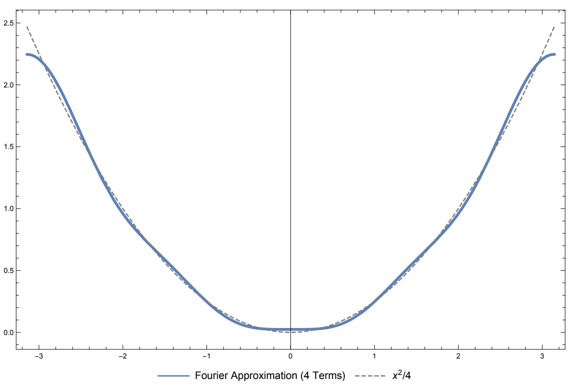


FIGURE 42.1. An illustration of the approximation of $f(x)$ on the interval $[-\pi, \pi]$ using four Fourier terms.

Corollary 42.2. *Assuming the Fourier decomposition for $x^2/4$ converges to the function itself on $[-\pi, \pi]$, then:*

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{42.1}$$

PROOF SKETCH. Since we defined the Fourier decomposition on $[-\pi, \pi]$, we have every right to expect that:

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi)$$

by setting $x = \pi$. Note (again) that:

$$\cos(n\pi) = (-1)^n$$

Therefore we have:

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Subtract $\pi^2/12$ from both sides to obtain Eq. (42.1). $\square$

Remark 42.3. This is Basel Problem first solved by Euler in 1734 (non-rigorously first, then rigorously later). It is one of the early hints at the deep connections between Number Theory and Fourier (Harmonic) Analysis. A different proof using Fourier methods can be produced using Pareval's Theorem.

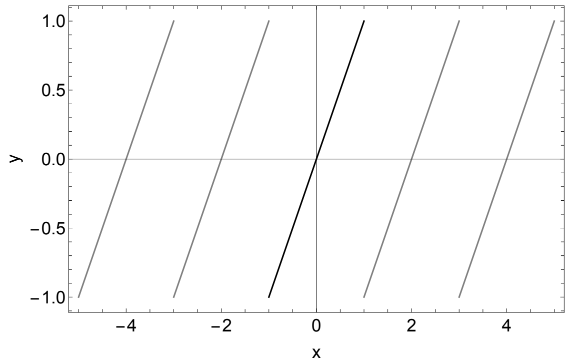


FIGURE 42.2. The periodic extension of the function $f(x) = x$ defined on the interval $[-L, L]$.

2. More on Fourier Sin and Cos Series

Remark 42.4. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* with period T if $f(x) = f(x+T)$ for all $x \in \mathbb{R}$. For example, the function $\sin(2\pi f x)$ is periodic with period $1/f$.

Definition 42.5 (Periodic Extension)**.** Suppose $f : [-L, L] \to \mathbb{R}$ is defined. The *periodic extension* of f to $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is the function defined as:

$$\tilde{f}(x) = f(x - 2Lm), \tag{42.2}$$

where m is the unique integer such that $-L \leq (x - 2Lm) \leq L$.

Remark 42.6. Definition 42.5 is a complicated way of saying we take f defined on the interval $[-L, L]$ and repeat it over and over in both directions. The same process can be repeated with any interval. For example $[0, L]$, but then we have to figure out what to do when we cross the origin. We illustrate this in Fig. 42.2 with the function $f(x) = x$ defined *only* on the interval $[-1, 1]$. We will see that using Fourier Sine or Cosine series decomposition for a function defined on $[0, L]$ has important consequences for its periodic extension outside this interval.

Lemma 42.7. *The sum of two even functions is even. The sum of two odd functions is odd.* □

Lemma 42.8. *The product of two even functions is even. The product of two odd functions is even. The product of an even and odd function is odd.* □

Lemma 42.9. *If $f(x)$ is odd and integrable on $[-L, L]$, then:*

$$\int_{-L}^{L} f(x)\, dx = 0.$$

PROOF. We have:

$$\int_{-L}^{L} f(x)\, dx = \int_{-L}^{0} f(x)\, dx + \int_{0}^{L} f(x)\, dx =$$
$$\int_{0}^{L} f(-x)\, dx + \int_{0}^{L} f(x)\, dx = -\int_{0}^{L} f(x)\, dx + \int_{0}^{L} f(x)\, dx = 0.$$

□

Lemma 42.10. *If $f(x)$ is even and integrable on $[-L, L]$, then:*

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

Remark 42.11. The following proposition follows from Lemmas 42.8 to 42.10 and Lemmas 40.2 and 40.9.

LESSON 43

1. Fourier Decomposition of Odd and Even Functions

Proposition 43.1. *If $f(x)$ is even, then $b_n = 0$ for all n in its Fourier decomposition on $[-L, L]$ and $f(x)$ is represented by a* <u>*Fourier cosine series*</u> *whose coefficients may be determined by Lemma 40.9. If $f(x)$ is odd, then $a_0 = a_n = 0$ for all n in its Fourier decomposition on $[-L, L]$ and $f(x)$ is represented by a* <u>*Fourier sine series*</u> *whose coefficients may be determined by Lemma 40.2.* □

Remark 43.2. What this means is that when we compute a Fourier sine series of a function on $[0, L]$, we will always recover its periodic extension as an *odd* function. When we compute a Fourier cosine series of a function on $[0, L]$ we will recover its even extension.

Example 43.3. Consider the odd function $f(x) = x$ and compute its Fourier cosine series on $[0, \pi]$ (here $L = \pi$). We have:

$$a_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)\,dx = \frac{2\,(-1+\cos(n\pi))}{\pi n^2} = \frac{2}{\pi n^2}\,(-1+(-1)^n) = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

On $[0, L]$ (and only on this interval) we have:

$$x \sim \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4\cos[(2n-1)x]}{\pi(2n-1)^2}, \tag{43.1}$$

which perfectly reproduces $f(x) = x$, see Fig. 43.1 (left). However, if we look at the plot of this approximation outside the region we see that the Fourier cosine series actually produces an even sawtooth wave, which models $f(x) = |x|$ on $[-\pi, \pi]$. See Fig. 43.1(right). This is a result of using a Fourier cosine series on an odd function.

Remark 43.4. We can obtain another remarkable series representation from the previous example. Set $x = 0$ in the Fourier cosine expansion in Eq. (43.1) to obtain:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

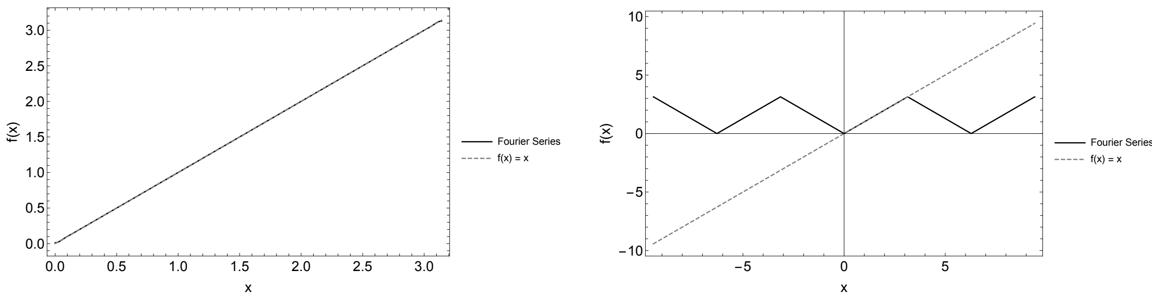


FIGURE 43.1. The Fourier cosine series of an odd function produces an even periodic extension, as expected.

Simplifying a bit yields:

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots.$$

Remark 43.5. In *many* books, when dealing with a Fourier cosine series, by convention, the series is written:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

If this is the case, then you can use the same formula for a_0 as a_n because you will be dividing by 2. (See Lemma 40.9.)

2. Convergence

THEOREM 43.6. *If $f(x)$ is piecewise C^1 on the interval $[-L, L]$ then the Fourier series $\hat{f}(x)$ of $f(x)$ converges to:*

(1) *The periodic extension of $f(x)$ on $\mathbb{R}$ where ever the periodic extension is continuous, or*
(2) *The average:*

$$\hat{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right],$$

if x is a point of discontinuity (i.e., a jump discontinuity). Here:

$$f(x^+) = \lim_{a \to x^+} f(a)$$
$$f(x^-) = \lim_{a \to x^-} f(a)$$

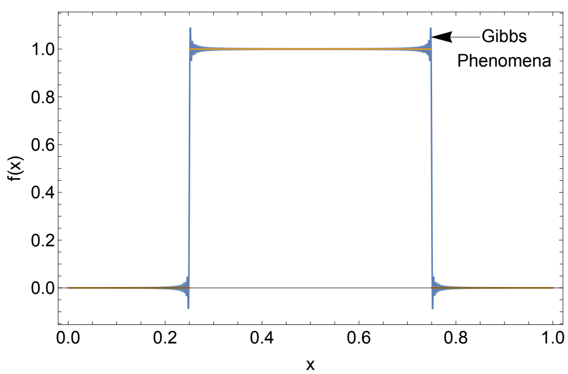
Remark 43.7. You will note, we are being *intentionally vague* about what we mean by *convergence*. In fact we will not even commit to whether that convergence is uniform or pointwise since it turns out it's dependent on the function itself.

Remark 43.8. The *shortest* proof of a form of this theorem is most likely in Rudin [**R+64**]. It's only a paragraph long, but relies on a number of auxiliary results. Haberman [**Hab03**] eschews a proof, focusing on the applications. Olvers [**Olv14**] has an exceptionally detailed proof. There is also a proof in Asmar [**Asm16**].

There are a few ways to approach the problem of proving this result (which we will not do). Olver's approach takes a nice detour through Hilbert space. Rudin's approach ignores the possibility of jump discontinuities and deals with the theorem as a part of a discussion on sequences and series of functions. It is a little less general, but a really beautiful proof.

The better part of the 150 years after Fourier lived and died saw mathematicians trying to quantify the conditions under which Fourier series converge and how they converge. Most of modern analysis is built upon and relies on these results. For our purposes, it suffices to think of convergence as being *close enough* for practical physics or engineering purposes.

Example 43.9 (Gibb's Phenomena)**.** In Theorem 43.6, we quantify the conditions under which a Fourier series will converge to the function it is approximating, but infinite sums are impossible to compute in practice. Therefore, it is worth understanding what happens when we use a finite sum to approximate a function. We have already seen this in Figs. 40.1 and 40.2. It is tempting to think this is simply a matter of using a short series approximation, but the overshoot seen near the jump discontinuity (approximately a 9% error) persists even as we add terms as shown in Fig. 43.2. This error is called Gibbs phenomena (named for J.



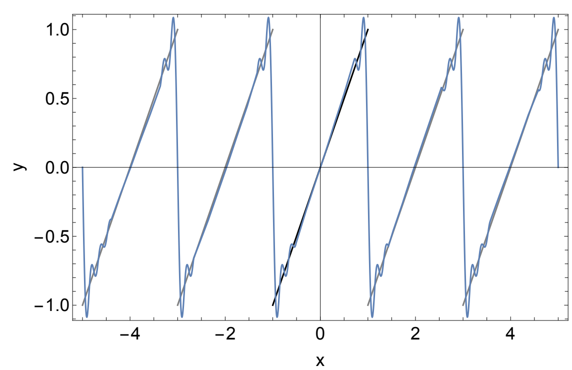


FIGURE 43.2. (Left) Gibbs Phenomena is persistent even as the number of terms in the Fourier series increases. This over/undershoot is about 9%. (Right) Gibbs phenomena presents as a result of discontinuities in the periodic extension of the function being approximated.

Willard Gibbs, an American physicist). It is present anytime there is a jump discontinuity in the periodic extension of the function being approximated. There are several explanations for the emergence of this phenomena but the most straightforward one is the fact that the Fourier series does *not* converge uniformly to the periodic extension of the underlying function because of the presence of the jump discontinuities.

LESSON 44

1. What is a Fourier Transform

Definition 44.1 (Inner Product on Function Spaces)**.** Let $f, g : [a, b] \to \mathbb{R}$ be two functions defined on the interval $[a, b]$. Then the *inner product* of these functions is:

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx,$$

provided it exists. If $f, g : [a, b] \to \mathbb{C}$ then the *(complex) inner product* is:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx,$$

provided it exists. Here $\overline{g(x)}$ is the complex conjugate of $g(x)$.

Definition 44.2 (Orthogonal Functions)**.** Two functions $f, g : [a, b] \to \mathbb{R}$ (respectively $f, g : [a, b] \to \mathbb{C}$) are *orthogonal* if $\langle f, g\rangle = 0$.

Corollary 44.3. *If $n \neq m$ are integers, then $\sin(n\pi x/L)$ and $\sin(m\pi x/L)$ are orthogonal on the interval $[-L, L]$. The same also holds for $\cos(n\pi x/L)$ and $\cos(m\pi x/L)$ for $m \neq n$ and $\sin(m\pi x/L)$ and $\cos(n\pi x/L)$ irrespective of m and n.* □

Remark 44.4 (What is a Fourier Decomposition?)**.** Recall from Vector Calculus that in $\mathbb{R}^2$, any vector $\vec{\mathbf{x}}$ can be written as:

$$\vec{\mathbf{x}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}},$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the standard basis vectors. Using the dot product we note that also that $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ because the vectors are orthogonal. (The dot product of orthogonal vectors is zero.) The coefficients (a, b) are the coordinates of the vector written in the standard basis. The same is true of the Fourier coefficients. We imagine the function $f : [-L, L] \to \mathbb{R}$ as a vector in an infinite dimensional space. The functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ are basis vectors in this space. The Fourier decomposition is just an expression of the function $f(x)$ in this basis. Note:

$$\vec{\mathbf{x}} \cdot \hat{\mathbf{i}} = a$$
$$\vec{\mathbf{x}} \cdot \hat{\mathbf{j}} = b$$

By taking a dot product of the vector $\vec{\mathbf{x}}$ with the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, we recover the coefficients. This is exactly what Eqs. (39.5) to (39.7) are doing – with an appropriate rescale because $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ are not unit vectors, as we showed in Lemmas 39.6 and 39.7.

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