Partial Differential Equation: Penn State Math 412 Lecture Notes

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Using These Notes

Welcome to the tenth set of course lecture notes I've written. While other students maintain I just like writing books and not getting paid for them, I still assert this is not a book. You need a book! Go away and come back when you have a real textbook. Okay, do you have a book? Alright, let's move on then. This is a set of lecture notes for Math 412–Penn State's undergraduate Partial Differential Equations Course. You read that correctly. In the course listings, Math 412 is titled "Fourier Series and Partial Differential Equations" but should really be called "Partial Differential Equations (with some Fourier Series)." Since I use these notes while I teach, there may be typographical errors that I noticed in class, but did not fix in the notes. If you see a typo, send me an e-mail and I'll add an acknowledgement. There may be many typos, that's why you should have a real textbook.

The lecture notes are loosely based on a number of books that are quite good. (Did I mention you should have a book?) I recommended Haberman's *Applied Differential Equations* to the class, because that's the book I used when I took this course a long time ago. That said, parts of that book can be very dense. Logan's *Applied Differential Equations* has a really good first chapter and is a bit more modern. If you like classics, Strauss' book *Partial Differential Equations* is also a good reference, but I always found it a bit dense – in a good way. Olver's *Introduction to Partial Differential Equations* is thicker than the others and dives right into first order partial differential equations, covering shock waves very early and the wave equation first (rather than doing the heat equation first). It is definitely geared more toward students who tend toward pure math. Also, in the "classics" category Asmar's *Partial Differential Equations* is encyclopedic and definitely feels very much motivated by physics or engineering, since it dedicates entire chapters to polar or spherical versions of PDE's. Finally, if you just want to get to the point, try Farlow's underrated *Partial Differential Equations for Scientists and Engineers*. It has 45 well written short chapters that teach methods, but very little theory. If you want the graduate school experience, then you should own Evan's *Partial Differential Equations*.

In order to use these notes successfully, you should have taken Calculus I - IV (Math 140, 141, 230 and 250 at Penn State), covering Vector Calculus (Marsden and Tromba is best especially when used with Div, Grad, Curl and All That) and Elementary Ordinary Differential Equations (Boyce and DiPrima is a good book). I will briefly review some of the information you need from these courses, but it's not going to be a substitute for having covered the material. That being said, I hope you enjoy using these notes.

CHAPTER 1

A Tour of Partial Differential Equations

The goals of this chapter are to introduce everyone to the basic vocabulary that surrounds partial differential equations and to give some example solutions and physical intuition behind them. We will prove some simple facts about example partial differential equations and discuss initial and boundary conditions.

1. Partial Differential Equations

Remark 1.1. The goal of this chapter is to introduce terms and examples that are relevant to the rest of the class. The worst thing in an (applied) math class is to worry about jargon, so we'll try to get it out of the way early.

Definition 1.2. A *partial differential equation* (PDE) is an equation involving an *unknown* function of several variables $u(x_1, \dots, x_n)$ with $n > 1$ and any number of its partial derivatives.

Remark 1.3 (Notation)**.** Notation for partial derivatives varies by text convenience. For simplicity, let $u(x, t)$ be a function of two variables. In this case, x is usually a *spatial* variable and t is usually a *temporal* (time) variable. By way of notation for derivatives we have the following equivalent expressions:

$$\frac{\partial u}{\partial t} = u_t = \partial_t u$$
$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \partial_{xx} u$$

Notation varies by author and whatever happens to be convenient. There are still other notations that can be used.

Example 1.4 (Simple Transport Equation)**.** Let $u(x, t)$ be a function of space x and time t and let $c \in \mathbb{R}$, then the equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1.1}$$

is a partial differential equation, which we'll refer to as the *simple transport equation* because there are ways to make it more complex and thus more general.

Remark 1.5. It's now worth asking, "Where do the solutions to PDE's live?" That is, what kinds of functions will we be looking at. Many (but not all) solutions will live in the C^k spaces.

Definition 1.6 (Differentiability Classes)**.** A function $u : D \subset \mathbb{R}^n \to \mathbb{R}$ is C^k if all partial derivatives of u up to order k exist and the functions resulting from differentiation are continuous. That is for all $j \leq k$:

$$v(x_1, \dots, x_n) = \frac{\partial^j u}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}} \tag{1.2}$$

exists and is continuous when $m_1 + \cdots + m_n = j$. If a function has all derivatives (and they are continuous), then the function is C^∞ or *smooth*.

Remark 1.7. Some books, including [Hab03] define *smooth* to mean continuous and differentiable. Others require the function to be continuous and twice differentiable to be smooth. In these notes, we'll give the specific C^k class as needed.

Example 1.8. The function:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is C^1. To see this, note that:

$$f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

is continuous but $f''(0)$ is not continuous and (in fact) doesn't even exist at $x = 0$. This is illustrated in Fig. 1.1.

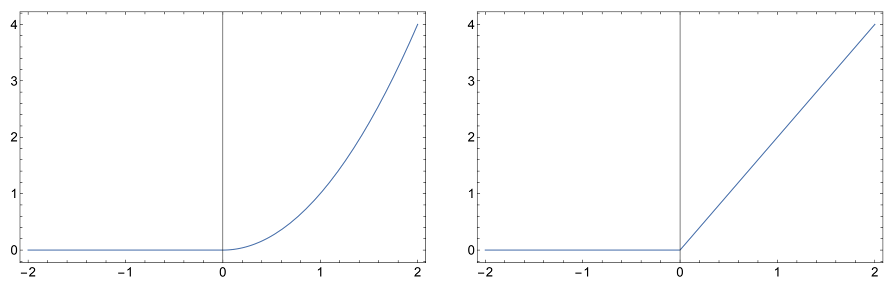

FIGURE 1.1. A C^1 function is illustrated whose derivative is C^0, that is continuous but without a continuous derivative.

Remark 1.9. It's worth noting that $C^0 \supset C^1 \supset C^2 \supset \cdots$, so if a function is C^k it is necessarily C^j for $j < k$.

Proposition 1.10. *Let $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^1$. Then $u(x,t) = f(x - ct)$ is a solution to the simple transport equation.*

PROOF. We can differentiate and check:

$$\frac{\partial u}{\partial t} = \frac{d}{dt} f(x - ct) = -cf'(x - ct).$$

On the other hand:

$$\frac{\partial u}{\partial x} = \frac{d}{dx} f(x - ct) = f'(x - ct).$$

Therefore:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -cf'(x - ct) + cf'(x - ct) = 0$$

□

Remark 1.11. Before commenting on the previous proposition, recall that the solution to the ordinary differential equation (ODE):

$$\frac{du}{dt} = \lambda u \tag{1.3}$$

is:

$$u(t) = Ae^{\lambda t},$$

where A is a constant of integration to be determined by an initial condition on the ODE.

By contrast, when we solved the simple transport equation, we don't just have an unspecified constant we have an unspecified function, namely $f(z)$. We can determine which C^1 function to use by imposing an initial condition.

Definition 1.12 (Initial Condition)**.** In a partial differential equation with n spatial variables $x_1, \dots, x_n$ and one time variable t, an initial condition (IC) is a requirement (constraint) of the form:

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = f(x_1, \dots, x_n). \tag{1.4}$$

This condition specifies the behavior of the unknown function (when $k = 0$) or one of its time derivatives (velocity, acceleration etc.) at the beginning of time.

Remark 1.13. While Definition 1.12 refers to time in order to make the language and the symbols consistent, we do not need to have a time variable at all to have an "initial condition." An initial condition is really just a statement that specifies the behavior of the unknown function $u(x_1, \dots, x_n)$ or ones of its derivatives when one of its variables is fixed at a specified value. In physical problems, most often, the initial condition specifies the behavior of the unknown function at the (perhaps arbitrary) beginning of time.

Corollary 1.14 (Corollary to Proposition 1.10)**.** *Consider the simple transport equation with the initial condition:*

$$u(x, 0) = f(x). \tag{1.5}$$

Then the solution $u(x, t) = f(x - ct)$ is a solution to the PDE that satisfies the initial condition. □

Remark 1.15. If the difference between Corollary 1.14 and Proposition 1.10 seems academic (pedantic). The difference is simply that in the corollary we have specified the function $f(x)$ defining the initial condition from the outside.

Exercise 1. Find a solution to the following PDE and IC:

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$$
$$u(x, 0) = f(x).$$

Illustrate your solution works. [Hint: The sign of c changed in the PDE. What do you think you should do to the solution from Corollary 1.14?]

Example 1.16 (Physical Interpretation of the Simple Transport Equation)**.** Let us be very concrete for a moment. Suppose we have the initial condition:

$$u(x,0) = e^{\frac{-x^2}{2}}.$$

Furthermore, suppose $c = 1$. The solution is:

$$u(x,t) = e^{\frac{-(x-t)^2}{2}}. \tag{1.6}$$

This is illustrated in Fig. 1.2. We see at once that the transport equation describes a function

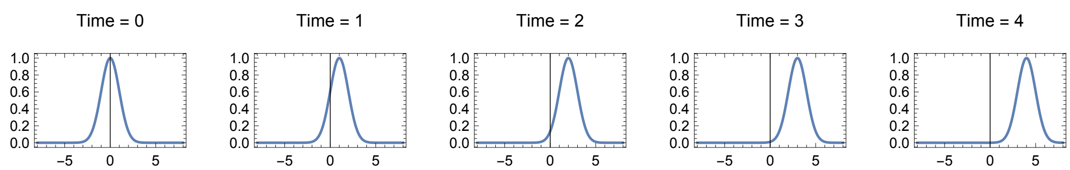

FIGURE 1.2. Solutions to the transport equation are functions traveling right.

$f(x)$ (denoting some physical system) moving to the right with speed c. In the example, the top of the curve is always at $x = t$, thus the curve is moving at speed $c = 1$ (as expected).

EXERCISE 2. What is the physical interpretation of the PDE with IC in Exercise 1?

EXERCISE 3. In Example 1.16 the assertion that c is the speed of motion is argued from an observation of a graph. Show that this is true by assuming that x is a function of t and is the position of an observer moving along with $u(x[t], t)$. Find the *total derivative* of $u[x(t), t]$ and observe that if the observer is moving with $u[x(t), t]$, then:

$$\frac{du}{dt} = 0.$$

Conclude c in the simple transport equation must be the speed of motion.

Remark 1.17. Given the physical interpretation of the simple transport equation, it's worth asking, "Does $u(x,t)$ have to be C^1?" The answer is, "no" but you'll have to change your definition of *differentiable* a little bit. In so doing, it would be perfectly acceptable to have the solution (e.g.):

$$u(x,t) = \begin{cases} 1 & \text{if } 0 < x - ct < 1 \\ 0 & \text{otherwise.} \end{cases}$$

This function is clearly not differentiable everywhere. However, it can be argued there is a way to define *weak differentiability* so that $u(x,t)$ is a perfectly reasonable solution that models a block of area 1 moving rightward along the x-axis at speed c. This generalization leads to the idea of Sobolev spaces, which are covered in graduate PDE's. This is the last time we will mention them. For the rest of the course, we're going to take a somewhat *relaxed* view of differentiability in the interest of finding physical meaning in our equations.

2. The Heat Equation & Initial Conditions

Definition 1.18 (Order of a PDE)**.** The *order* of a PDE is the highest order derivative that appears in the PDE.

Definition 1.19. Laplacian The *Laplacian* or *Laplace Operator* of a function $u \in C^2$ (or generalization) is:

$$\nabla \cdot (\nabla u) = \nabla^2 u = \Delta u. \tag{1.7}$$

Here ∇u is the *gradient* of the function u, which gives a vector field and the dot product of ∇ with a vector field gives the divergence, so:

$$\nabla^2 u = \operatorname{div}(\operatorname{grad}[u])$$

In Cartesian coordinates this is:

$$\nabla^2 u = \sum_i \frac{\partial^2 u}{\partial x_i^2}.$$

Remark 1.20. The Laplace operator is different depending on the coordinate system being used. We will discuss the variations when we need them. It is also worth noting that the Laplacian always refers to the spatial variables and not the temporal variable. So if there is a function with both spatial variables and a temporal variable, the Laplacian only differentiates with respect to the spatial variables unless otherwise indicated.

Remark 1.21. The simple transport equation is a first order PDE because it only contains first order derivatives. We contrast this with the heat equation.

Example 1.22 (Heat (Diffusion) Equation)**.** Let $k > 0$. The one-dimensional *heat (or diffusion) equation* is the second order partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{1.8}$$

In two or more dimensions, we modify the equation as:

$$\frac{\partial u}{\partial t} = k\nabla^2 u = k\Delta u. \tag{1.9}$$

Proposition 1.23. *Assume $t \in (0, \infty)$ and $x \in (-\infty, \infty)$, then the function:*

$$\Phi(x,t) = \frac{e^{-\frac{x^2}{4kt}}}{2\sqrt{k\pi t}} = \frac{1}{2\sqrt{k\pi t}} \exp\left(-\frac{x^2}{4kt}\right) = \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi k t}} \tag{1.10}$$

solves the heat equation in 1 dimension.

EXERCISE 4. Prove the previous proposition. [Hint: Differentiate in t, then differentiate twice in x. Don't forget to multiply by k. Feel free to use Mathematica, Maple, Wolfram Alpha or your TI-89/TI-Inspire or any other computer algebra system.]

Remark 1.24. See Example 1.72 for more on the Heat Kernel.

Corollary 1.25. *The solution $\Phi(x,t)$ for the heat equation decays to zero as time goes to infinity; i.e.,*

$$\lim_{t\to\infty} \Phi(x,t) = 0.$$

□

Example 1.26 (Physical Interpretation of the Heat Equation)**.** At this point, it's probably best to get used to weird functions. Imagine an infinitely powerful match is placed on a very thin rod (line) that extends in both directions forever. This is illustrated in Fig. 1.3. Assume

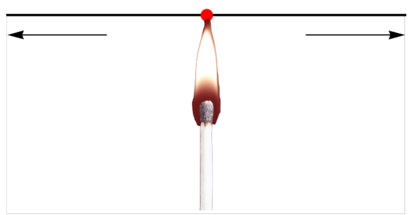

FIGURE 1.3. An infinitely strong match heats up the very center of an infinitely long line. Eq. (1.10) describes the evolution of this heat over time.

the match is instantaneously removed so that at time $t = 0$, the point $x = 0$ is infinitely hot and everywhere else on the line the temperature is zero. We have:

$$u(x,0) = \delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.11}$$

Then the function $\Phi(x,t)$ describes the *diffusion* of heat into the rod over time. Because the rod is infinite, ultimately all the heat diffuses into the rod and the temperature everywhere returns to zero (as time goes to infinity). This is shown in Fig. 1.4. Our goal will be to prove results like this as the course goes on.

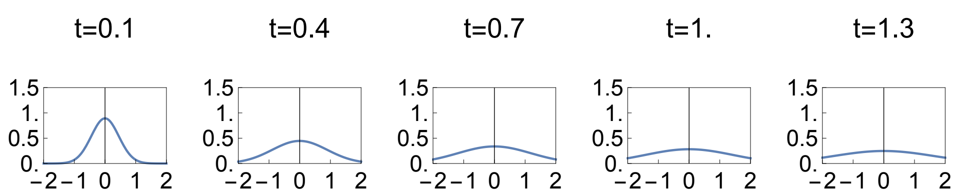

FIGURE 1.4. The heat equation models the diffusion of heat in an object.

Remark 1.27. The function $\Phi(x,t)$ is sometimes called the *heat kernel* because it's a building block for more general solutions to the heat equation. You may also recognize it from probability theory. It is a Gaussian curve. We'll come back to this later in the course.

Definition 1.28 (Dirac Delta)**.** The function

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

is the *Dirac delta* generalized function named for physicist Paul A. M. Dirac who (among other things) derived the relativistic Shrödinger equation.

Remark 1.29. You've seen the Dirac Delta function in your ODE class. However, it's worth pointing out what a strange object this is. It is not considered a classical *function* but instead a generalized function or *distribution*.

EXERCISE 5. The constant k in the heat equation is sometimes called the diffusion constant or diffusivity constant. How does the heat constant affect the diffusion of heat? That is if given two materials with different diffusion constants k_1 and k_2 with $k_2 > k_1$, which material conducts heat better?

EXERCISE 6. When studying the heat equation, mathematicians often set $k = 1$. However, the actual value of k is critical for engineering problems. Show that for any solution $u(x,t)$ of the heat equation (with some k), we can convert it to a solution of the heat equation when $k = 1$. Thus (in mathematics) it suffices to set $k = 1$. Hint: Let $v(x,t) = u(x, t/k)$. Show the resulting function solves:

$$\frac{\partial v}{\partial t} = \nabla^2 v. \tag{1.13}$$

Interpret this in light of your answer to the previous exercise.

EXPERIMENT 1. You probably have neither an infinitely long rod nor an infinitely powerful match lying around your house, but you can simulate this experiment. Poke a spoon through a paper towel (or cut a small hole in a piece of paper and insert a spoon through it). Let that sit while you boil some water and pour it into a mug. Place the spoon in the mug with the paper over the mug so that the handle of the spoon is sticking out of the mug and the paper towel is keeping the (hot) water vapor in the mug. Wait a few minutes. The handle of spoon should now be warm because of heat transfer from the water through the metal.

Remark 1.30. Understanding a physical interpretation of the Laplacian, can help make sense of the heat equation and its properties.

Proposition 1.31. *Assume $\epsilon > 0$ and $\epsilon \ll 1$. Let $u(x,t)$ be a C^2 function. Then:*

$$u_{xx}(x,t) \approx \frac{2}{\epsilon^2}\left[\frac{u(x+\epsilon,t) + u(x-\epsilon,t)}{2} - u(x,t)\right] \tag{1.14}$$

PROOF. Using a modified Newton quotient, we have the approximation:

$$u_x(x,t) \approx \frac{u\left(x + \frac{1}{2}\epsilon, t\right) - u\left(x - \frac{1}{2}\epsilon, t\right)}{\epsilon}. \tag{1.15}$$

It follows that:

$$u_{xx}(x,t) \approx \frac{u_x\left(x + \frac{1}{2}\epsilon, t\right) - u_x\left(x - \frac{1}{2}\epsilon, t\right)}{\epsilon}. \tag{1.16}$$

Substitute Eq. (1.15) in Eq. (1.16) to obtain:

$$u_{xx}(x,t) \approx \frac{1}{\epsilon}\left[\frac{u\left(x + \frac{1}{2}\epsilon + \frac{1}{2}\epsilon, t\right) - u\left(x + \frac{1}{2}\epsilon - \frac{1}{2}\epsilon, t\right)}{\epsilon} - \right.$$
$$\left. \frac{u\left(x - \frac{1}{2}\epsilon + \frac{1}{2}\epsilon, t\right) - u\left(x - \frac{1}{2}\epsilon - \frac{1}{2}\epsilon, t\right)}{\epsilon}\right].$$

Simplifying yields:

$$u_{xx}(x,t) \approx \frac{1}{\epsilon^2}\left\{[u(x+\epsilon) - u(x)] - [u(x) - u(x-\epsilon)]\right\} = \frac{1}{\epsilon^2}\left\{[u(x+\epsilon) + u(x-\epsilon)] - 2u(x)\right\}.$$

Factoring out a 2 from the right-hand-side yields:

$$u_{xx}(x,t) \approx \frac{2}{\epsilon^2}\left[\frac{u(x+\epsilon,t) + u(x-\epsilon,t)}{2} - u(x,t)\right].$$

This completes the proof. □

Remark 1.32. Proposition 1.31 allows us to interpret the Laplacian. Notice:

$$\frac{u(x+\epsilon,t) + u(x-\epsilon,t)}{2}$$

is the *arithmetic mean* of the function u around the point x (see Fig. 1.5). Thus,

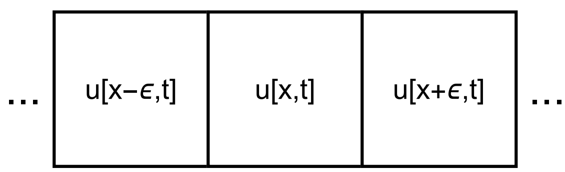

FIGURE 1.5. The Laplacian effectively compares the value of $u(x,t)$ with the mean of the values around $u(x,t)$.

$$u_{xx} \sim \frac{u(x+\epsilon,t) + u(x-\epsilon,t)}{2} - u(x,t) > 0$$

if and only if $u(x,t)$ is *smaller* than the average of its surrounding values. On the other hand,

$$u_{xx} \sim \frac{u(x+\epsilon,t) + u(x-\epsilon,t)}{2} - u(x,t) < 0$$

if and only if $u(x,t)$ is *bigger* than the average of its surrounding values. Returning to the heat equation:

$$u_t = k u_{xx},$$

implies that if $u(x,t)$ is *bigger* than the average of the surrounding values, then $u_t < 0$ and $u(x,t)$ gets smaller. Conversely, if $u(x,t)$ is *smaller* than the average of the surrounding values, then $u_t > 0$ and $u(x,t)$ gets bigger. Thus, the heat equation describes a smoothing process in which "big" values spread toward areas with "smaller" values.

Remark 1.33. It is worth noting that this interpretation of the Laplacian extends to the higher dimensional Laplacian as well.

Example 1.34 (Schrödinger Equation)**.** Related to, but distinct from, the heat equation is the Schrödinger Equation of quantum mechanics:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi \tag{1.17}$$

This is a diffusion equation with an *imaginary* diffusion constant (recall $i = \sqrt{-1}$). This small change makes a big difference in the nature of the solutions. Also, unlike the heat equation, which we will derive in the next chapter, Schrödinger's Equation does not seem to have a natural derivation; it simply popped out of Schrödinger's mind. [Note: There are several variations of the Schrödinger Equation. Eq. (1.17) assumes zero potential and is written in the position basis. If you're really interested in the Schrödinger Equation, try [**Fle20**].]

3. Laplace Equation & Boundary Conditions

Remark 1.35. So far, we have considered PDE's on infinite domains like $(-\infty, \infty)$. For the heat equation, this is non-physical since we do not have infinitely long rods, or infinitely large sheets (of metal). We can fix this problem by introducing boundary conditions.

Definition 1.36 (One Dimensional Dirichlet Boundary Condition)**.** Consider a PDE with unknown function $u(x, t)$. Suppose we are interested in solutions to this PDE on the interval $[0, L]$. A *Dirichlet Boundary Condition* has form:

$$\forall t \begin{cases} u(0,t) = a \\ u(L,t) = b \end{cases}$$

That is, it fixes the values of the unknown function at the ends of the interval. If $a = b = 0$, then the boundary conditions are called *homogeneous*.

Remark 1.37. Some books (notably [**Olv14**]) allow a and b in Definition 1.36 to be functions of time. For now, we will not worry about this case, but it is worth noting it is possible and still falls under the name Dirichlet Boundary Condition.

Example 1.38. Consider the one-dimensional heat equation with initial condition $u(x, 0) = f(x)$ and boundary conditions:

$$u(0,t) = u(L,t) = 0$$

Then the fully specified PDE becomes:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} \\ \text{IC: } u(x,0) = f(x) \\ \text{BC: } u(0,t) = u(L,t) = 0 \end{cases}$$

A physical interpretation for this situation is shown in Fig. 1.6. Here we have a finite length,

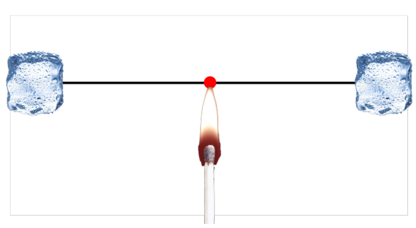

Figure 1.6. An illustration of a physical interpretation of the heat equation with homogeneous Dirichlet boundary conditions.

infinitely thin rod with its ends embedded in ice.

Definition 1.39 (General Dirichlet Boundary Condition)**.** Consider a PDE with unknown function $u(x_1, \dots, x_n, t)$ and suppose we are interested in the solution on the interior of a region $\Omega \subset \mathbb{R}^n$. Then a *Dirichlet boundary condition* has form:

$$u(x, t) = f(x) \quad \forall \mathbf{x} \in \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of the region Ω.

Example 1.40. We illustrate a region with its boundary in Fig. 1.7. The boundary is shown as a dashed line. The interior is shown in gray.

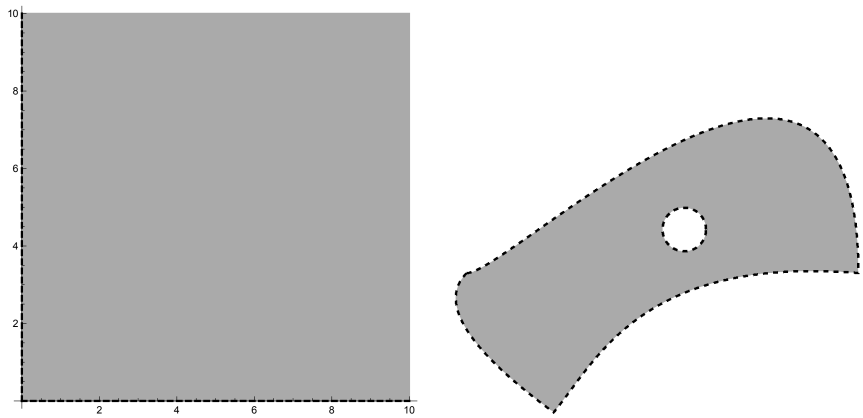

FIGURE 1.7. (Left) An infinite region bounded by two lines. (Right) A finite region with a hole. All boundaries are shown as dashed lines.

Remark 1.41. It is very much worth noting that we are being intentionally vague about the region Ω and its properties. In this course, we will restrict our attention to friendly regions like intervals in $\mathbb{R}$, disks, rectangles or annuli in $\mathbb{R}^2$ or perhaps a sphere in $\mathbb{R}^3$. Solving PDE's on more general regions is a topic best left for a more advanced class and often requires the use of numerical methods.

Definition 1.42 (One Dimensional Neumann Boundary Condition)**.** Consider a PDE with unknown function $u(x, t)$. Suppose we are interested in solutions to this PDE on the interval $[0, L]$. A *Neumann Boundary Condition* has form:

$$\forall t \begin{cases} u_x(0, t) = a(t) \\ u_x(L, t) = b(t) \end{cases}$$

That is, it fixes the values of the first spatial derivative of the unknown function at the ends of the interval.

Example 1.43. Consider the one-dimensional heat equation with initial condition $u(x, 0) = f(x)$ and Neumann boundary conditions:

$$u_x(0, t) = u_x(L, t) = 0$$

Then the fully specified PDE becomes:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} \\ \text{IC: } u(x, 0) = f(x) \\ \text{BC: } u_x(0, t) = u_x(L, t) = 0 \end{cases}$$

A physical interpretation for this situation is shown in Fig. 1.8. Here we have an infinitely

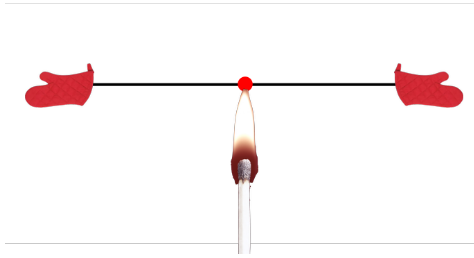

FIGURE 1.8. An illustration of a physical interpretation of the heat equation with homogeneous Neumann boundary conditions.

thin rod of length L that has perfect insulation. The homogeneous Neumann boundary condition specifies that the *heat flow* out of (or into) the rod is zero.

Definition 1.44 (General Neumann Boundary Condition)**.** Consider a PDE with unknown function $u(x_1, \dots, x_n, t)$ and suppose we are interested in the solution in a region $\Omega \subset \mathbb{R}^n$. Then a *Neumann boundary condition* has form:

$$\nabla u \cdot \mathbf{n}(\mathbf{x}) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega,$$

where $\mathbf{n}$ is a normal vector for the region Ω at point $\mathbf{x} \in \partial\Omega$.

Remark 1.45. Recall from vector calculus that $\nabla u \cdot \mathbf{n}(\mathbf{x})$ is the *directional derivative* of u at point $\mathbf{x}$ in the (normal) direction $\mathbf{n}$. A simple surface and some normal vectors are illustrated below. The idea is to specify the rate of flow of the quantity u into (or out of)

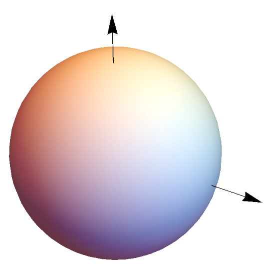

FIGURE 1.9. Some normal vectors to a sphere in $\mathbb{R}^3$.

the surface in question. In more than three dimensions, this becomes difficult to visualize and the definitions become slightly more complex, but the idea is the same.

Definition 1.46 (Laplace/Poisson/Helmholtz Equation)**.** Let $u(x_1, \dots, x_n)$ be a function defined only on spatial variables. Then the *Laplace Equation* is the second order partial differential equation:

$$\nabla^2 u = \Delta u = 0. \tag{1.18}$$

If we replace 0 with an arbitrary function $f(x_1 \dots, x_n)$, then we have the *Poisson Equation*, which is a natural generalization. When 0 is replaced by the unknown function λu, (for some λ) then the result is called the *Helmholtz Equation*. (Note some authors use $-\lambda u$ instead.)

Remark 1.47 (Physical Interpretation of the Laplace Equation)**.** Consider the heat equation on a region:

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

This is the unit disk. We can also write this in polar coordinates as:

$$\Omega = \{(r, \theta) : r \in [0, 1],\, \theta \in [0, 2\pi]\}$$

It is easier to define a non-homogeneous boundary condition using polar coordinates. Suppose it is:

$$u(1, \theta) = \sin(4\theta).$$

Note, we must automatically impose a *periodic boundary condition* namely:

$$u(r, 0) = u(r, 2\pi),$$

which is ensured at the boundary by the boundary condition given for the edge of the disk when $r = 1$. Note, we also require $u_\theta(r, 0) = u_\theta(r, 2\pi)$, but we will not use this boundary condition in practice.

If we imagine a *long time* has passed so that the disk has come to thermal equilibrium, then:

$$\frac{\partial u}{\partial t} = 0. \tag{1.19}$$

That is, there are only spatial variations in the temperature but it is no longer changing in time. This is called a *stationary solution* of the heat equation and since Eq. (1.19) holds, the function $u(x, y)$ (ignoring t) must satisfy the Laplace equation. (Substitute Eq. (1.19) into the heat equation.) The resulting solution is illustrated in Fig. 1.10. In the figure, we can

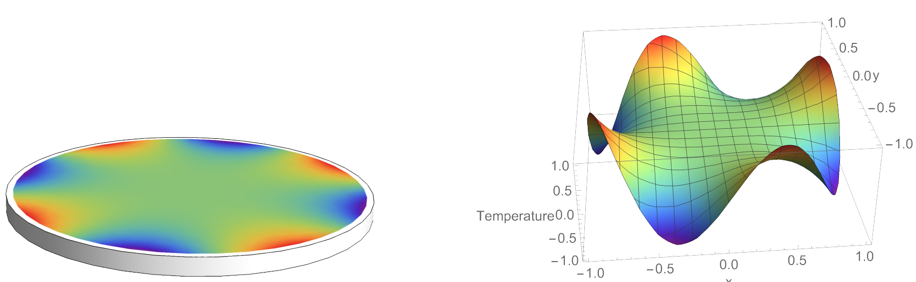

FIGURE 1.10. An illustration of the steady state solution of the heat equation on the disk assuming a non-homogeneous sinusoidal (temperature) boundary condition. The initial conditions are irrelevant.

see four red high temperature regions around the plate corresponding to the four maxima that occur in the function $u(1, \theta) = \sin(4\theta)$ as well as four blue cool temperature regions corresponding to the four minima that occur in the boundary condition. The remainder of the plate illustrates the temperature at thermal equilibrium.

Remark 1.48. Continuing the discussion from Remarks 1.32 and 1.33, we see exactly what Laplace's equation means: a function $u(\mathbf{x})$ satisfies Laplace's if it is locally equal to its mean value. This is precisely the nature of functions that describe equilibria.

Definition 1.49 (Periodic Boundary Conditions)**.** A boundary condition on a C^m function $u(x_1, \dots, x_n)$ is *periodic* if

$$\left.\frac{\partial^j u}{\partial x_1^j}\right|_{x_1=a_1} = \left.\frac{\partial^j u}{\partial x_1^j}\right|_{x_1=b_1} \tag{1.20}$$

$$\left.\frac{\partial^j u}{\partial x_2^j}\right|_{x_2=a_2} = \left.\frac{\partial^j u}{\partial x_2^j}\right|_{x_2=b_2} \tag{1.21}$$

$$\vdots \tag{1.22}$$

$$\left.\frac{\partial^j u}{\partial x_n^j}\right|_{x_n=a_n} = \left.\frac{\partial^j u}{\partial x_n^j}\right|_{x_n=b_n} \tag{1.23}$$

for $j = 0, \dots, m$.

Remark 1.50. It turns out that the Laplace equation in the plane (R^2) has connections to complex analysis. In particular, analytic functions (functions of a complex variable that are infinitely differentiable) yield solutions to the Laplace equation. This allows one to pull some solutions "out of thin air." For example,

$$u(x, y) = e^x \cos(y) \tag{1.24}$$

is a solution to the Laplace equation on the plane ($\mathbb{R}^2$) with no specified boundary conditions. This can be verified with differentiation.

Exercise 7. Verify that Eq. (1.24) is a solution to Laplace's equation.

Exercise 8. Use what you know about the derivative of $\sin(y)$ to come up with another solution to the Laplace equation.

Exercise 9. Show that $\log\left(\sqrt{x^2 + y^2}\right)$ is a solution to the Laplace equation in two spatial dimensions. (As it happens, up to a scaling factor, this is the fundamental solution to the Laplace equation in two dimensions.)

4. The Wave Equation & Classification of Second Order PDE's

Definition 1.51 (Wave Equation)**.** Let $c \in \mathbb{R}$. The one-dimensional *wave equation* is the second order partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.25}$$

In two or more dimensions, we modify the equation as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \Delta u. \tag{1.26}$$

Proposition 1.52. *Let $F(z)$ and $G(z)$ be C^2 functions (or some generalization). Then:*

$$u(x, t) = F(x - ct) + G(x + ct) \tag{1.27}$$

is a solution to the wave equation.

Proof. The proof is by differentiation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 F''(x - ct) + c^2 G''(x + ct)$$

and

$$\frac{\partial^2 u}{\partial x^2} = F''(x - ct) + G''(x + ct)$$

The result follows immediately. □

Remark 1.53. The solution given in Proposition 1.52 was derived by D'Almbert and forms the basis of a general solution (called D'Almbert's formula), which we will come to when we study the wave equation in detail. This formulation, unfortunately, is not exceptionally useful (in the form given) for cases with boundary or initial conditions. In the coming chapters, we will find ways to add these additional pieces of information to obtain general solutions to the wave equation.

Example 1.54 (Wave Equation Solution)**.** Let us return to the initial condition of Example 1.16 with

$$u(x, 0) = e^{\frac{-x^2}{2}}.$$

Then a legitimate solution of the wave equation satisfying this initial condition is:

$$u(x,t) = \frac{1}{2}e^{\frac{-(x-ct)^2}{2}} + \frac{1}{2}e^{\frac{-(x+ct)^2}{2}} \tag{1.28}$$

This solution is illustrated in Fig. 1.11. In this case, a solution to the wave equation models

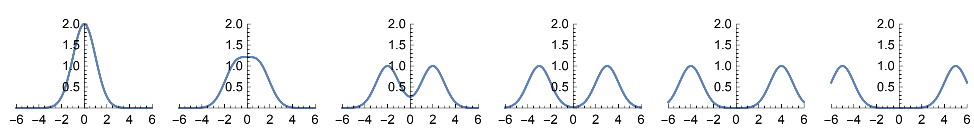

Figure 1.11. A solution to the wave equation with left and right moving waves.

left and right transverse (sideways) motion. The initial condition splits into two functions (each half the size of the original) that travel to the left and right.

Example 1.55 (Physical Interpretation of the Wave Equation)**.** As we will prove in future chapters, the wave equation models vibrations within a medium like a string or a drumhead. This is especially relevant when appropriate boundary conditions are imposed. By way of example, let Ω be the unit disk from Remark 1.47. Consider the PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$$\text{IC: } u(x,y,0) = 1 - x^2 - y^2 \quad u_t(x,y,0) = -1$$

$$\text{BC: } u(x,y,t) = 0 \quad \forall (x,y) \in \partial\Omega.$$

Notice we have a second initial condition: $u_t(x, y, 0) = -1$. This is equivalent to the second initial condition used in second order ordinary differential equations. It occurs because the second derivative of time is used in the wave equation (as compared to the first derivative in heat equation). Solving this problem is a task we will consider later in the semester, but a plot of the solution is shown in Fig. 1.12. The resulting solution shows a vibrating circular

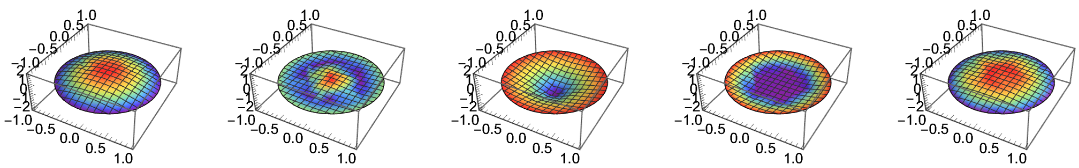

FIGURE 1.12. The solution to the wave equation with the Dirichlet boundary condition yields a solution that resembles the vibration of a membrane (drum) when it is stretched and then released with constant downward acceleration.

"membrane" (drum head) that will oscillate up and down forever (since no drag forces are modeled).

Remark 1.56. We have now encountered three distinct second order partial differential equations, each with a distinct function:

(1) The heat equation, which models diffusion,
(2) Laplace's equation, which models equilibria and
(3) The wave equation, which models waves.

These equations are representatives of the three classes of second order partial differential equations.

Definition 1.57 (Classes of Second Order PDE's [Far93])**.** Consider an arbitrary linear PDE with constant coefficients in variables, which we write as:

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + F = 0$$

or if using spatial variables:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$$

(1) If $B^2 - 4AC = 0$, then the equation is classified as *parabolic*.
(2) If $B^2 - 4AC > 0$, then the equation is classified as *hyperbolic*.
(3) If $B^2 - 4AC < 0$, then the equation is classified as *elliptic*.

EXERCISE 10. Convince yourself that the heat equation is parabolic, Laplace's equation is elliptic and the wave equation is hyperbolic.

EXERCISE 11. What kind of equation is the Poisson equation? [Hint: See Definition 1.46.]

Remark 1.58. When moving from one spatial and one temporal dimension (or two spatial dimensions) to higher spatial dimensions, it is still possible to classify linear equations with constant coefficients into parabolic, elliptic and hyperbolic classes. The formula for the discriminant (e.g., $B^2 - 4AC$) is just slightly more complex. See [Eva15] for details. The higher dimensional analogs to the heat, Laplace and wave equations are parabolic, elliptic and hyperbolic respectively; adding extra dimensions does not change the type of the equation (unless you change the structure of the equation in a fundamental way).

Remark 1.59. Recall Example 1.26 and the function $\Phi(x, t)$ from Proposition 1.23, which solves the heat equation. The information about the infinitely strong match has already passed to every point as soon as $t > 0$ because $\Phi(x, \epsilon) > 0$ for all $\epsilon > 0$. This is a property of parabolic equations as is the fact they model diffusion and their solutions *tend* to be smoother than their initial conditions (because of the inherent mixing that occurs as part of diffusion). On the other hand, information in the wave equation propagates at speed c, as we can see from Example 1.54. This is a property of elliptic equations as is the fact that they model wave like systems. Finally, the Laplace equation has no time information at all. Its solution describes an equilibrium state as do elliptic PDE's in general. Moreover, solutions to this equation cannot have discontinuities (they have been smoothed away as a property of equilibrium).

5. Linearity, Linear Operators & Homogeneous PDE's

Definition 1.60 (Linear Partial Differential Equation)**.** A PDE is *linear* if it is a linear function of the unknown function u and all of its derivatives. Otherwise, it is called *non-linear*.

Example 1.61. Linearity can be a tricky concept because it applies only to the unknown function and not the variables $x_1, \dots, x_n$ or t. The heat, wave and Laplace equations are all linear. However, the equations:

$$u_t + cxu_x + tu = 0$$
$$u_{tt} - c^2 u_{xx} = \cos(x - t).$$

are still linear. You will notice in the first equation, it is perfectly acceptable to have coefficients for the unknown function u and its derivatives that are functions of the independent variables (x and t in this case). In the second example the non-linearity – i.e., $\cos(x - t)$ – only involves the independent variables and not the unknown function u or its derivatives. Consequently, these equations are linear.

By contrast, the partial differential equation (known as *Burger's equation*):

$$u_t + uu_x = \nu u_{xx}$$

is *nonlinear* because it contains a product of the unknown function u with one of its derivatives u_x. We will come to solutions of Burger's equation (and their unusual properties) later in the course.

Remark 1.62. In the next definition we're going to use the term *space*, which is quite generic. While this is not exclusively true, the term space usually means a *vector space* or *function space*; that is a set of vectors or functions (which can sometimes be treated as vectors) that is closed under addition and multiplication by a real (or complex) number and has some other useful properties. As we move forward, we'll keep the discourse relatively high-level and not worry too much about the details.

Definition 1.63 (Linear Operator)**.** An *operator* is a mapping from a space U to a space V. Given two elements $u, v \in U$ (resp. V), assume that $u + v \in U$ (resp. V) and for some $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$) then $\alpha u \in U$ (resp. V). An operator $L : U \to V$ is *linear* if:

$$L(\alpha u + v) = \alpha L(u) + L(v) \tag{1.29}$$

Example 1.64. You already know two linear operators that are critical in the study of calculus. The derivative and the integral are both linear operators. To see this, recall the following rules from calculus. Assume $f, g : \mathbb{R} \to \mathbb{R}$ are C^1 functions, then:

$$\frac{d}{dx}(f + \alpha g) = \frac{df}{dx} + \alpha \frac{dg}{dx}.$$

Likewise, if $f, g : \mathbb{R} \to \mathbb{R}$ are C^0 functions, then:

$$\int \alpha f(x) + g(x)\, dx = \alpha \int f(x)\, dx + \int g(x)\, dx$$

Example 1.65. We can generalize the idea of an operator to re-write linear PDE's in simpler form. Consider the simple transport equation and define:

$$L = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

Then for some C^1 function u, we have:

$$L(u) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}.$$

Thus, the simple transport equation can be written as:

$$Lu = 0.$$

Example 1.66 (D'Almbertian)**.** The D'Almbertian operator defines the wave equation. In one dimension we have:

$$\Box = \frac{\partial}{\partial t^2} - c^2 \frac{\partial}{\partial x^2}.$$

In higher dimensions it is:

$$\Box = \frac{\partial}{\partial t^2} - c^2 \Delta.$$

The wave equation is then:

$$\Box u = 0.$$

It is worth noting that in one dimension, you can factor the D'Almbertian as:

$$\Box = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right).$$

This helps to explain D'Almbert's solution to the wave equation as a pair of traveling waves, one going leftward and another going rightward.

Exercise 12. Find the "diffusion (heat) operator" and write the heat equation as an operator applied to a function u.

Proposition 1.67. *Every linear PDE can be written in operator form:*

$$L(u) = g_0(x_1, \dots, x_n).$$

Remark 1.68. A general proof is notationally intensive and completely un-instructive for any equations above second order. Therefore, we'll provide a proof for second order PDE's with a single spatial variable and a single temporal variable.

Restricted Proof. Let:

$$F(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = -g_0(x,t)+g_1(x,t)u(x,t)+g_2(x,t)u_t(x,t)+g_3(x,t)u_x(x,t)+ g_4(x,t)u_{tt}(x,t) + g_5(x,t)u_{tx}(x,t) + g_6(x,t)u_{xx}(x,t).$$

Without loss of generality, we assume the PDE is expressed as:

$$F(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0.$$

Define:

$$L = \left(g_1(x,t) + g_2(x,t)\frac{\partial}{\partial t} + g_3(x,t)\frac{\partial}{\partial x} + g_4(x,t)\frac{\partial^2}{\partial t^2} + g_5(x,t)\frac{\partial^2}{\partial t \partial x} + g_6(x,t)\frac{\partial^2}{\partial x^2}\right).$$

Then we can rewrite the PDE as:

$$L(u) = g_0(x,t).$$

This completes the proof. □

Exercise 13. Explain why we can make that assumption without loss of generality in the proof.

Exercise 14. Consider the nonlinear PDE:

$$u_t + uu_x = 0$$

Explain why the trick used in the proof of Proposition 1.67 cannot be used here. That is what happens when you try to define a "simple" operator?

Definition 1.69. A linear partial differential equation is called *homogeneous* if the function $g_0 \equiv 0$. Otherwise, the PDE is called non-homogeneous.

Remark 1.70. Most of the PDE's we have encountered in this chapter are homogeneous. As we'll see in the next proposition, these PDE's are in a sense "nice" because they have algebraic structure. Moreover, solutions for non-homogeneous PDE's can often be constructed from solutions to homogeneous PDE's and a *specific* solution that takes into account the function g_0. By contrast, nonlinear PDE's are *not nice*. Solving these equation is often exceptionally hard[1].

Proposition 1.71. *Consider a linear homogeneous PDE:*

$$L(u) = 0,$$

defined by operator L. If u and v are two separate solutions defined on $\mathbb{R}^n$, then $\alpha u + v$ is a third solution for $\alpha \in \mathbb{R}$.

Proof. The proof is by direct construction. We have: $L(u) = L(v) = 0$. Then:

$$0 = \alpha L(u) + L(v) = L(\alpha u) + L(v) = L(\alpha u + v),$$

by linearity. □

[1]The Navier-Stokes equations are so hard, there is a \$1M prize on their head for an analytic solution or a proof that the equations are bounded and well-behaved (or an example of blow up).

Example 1.72 (The Heat Kernel)**.** Using Proposition 1.71 we see that exact form of the heat kernel:

$$\Phi(x,t) = \frac{e^{-\frac{x^2}{4kt}}}{2\sqrt{k\pi t}} = \frac{1}{2\sqrt{k\pi t}} \exp\left(-\frac{x^2}{4kt}\right)$$

is not unique. In fact, we could multiply the heat kernel by $\sqrt{2}$ to obtain:

$$\tilde{\Phi}(x,t) = \frac{e^{-\frac{x^2}{4kt}}}{2\sqrt{k\pi t}} = \frac{1}{\sqrt{2\pi kt}} \exp\left(-\frac{x^2}{4kt}\right),$$

which is still a solution to the heat equation. We could even add the two solutions together to obtain $\Phi(x,t) + \tilde{\Phi}(x,t)$, which is still a solution to the heat equation.

Corollary 1.73. *Consider the linear homogeneous PDE defined on $\Omega \subseteq \mathbb{R}^n$ with homogeneous boundary conditions:*

$$\begin{aligned} &L(u) = 0 \\ &\textit{BC: } u(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \partial\Omega \end{aligned}$$

then $\alpha u + v$ is a third solution for $\alpha \in \mathbb{R}$. □

Exercise 15. Add an extra observation to the proof of Proposition 1.71 to prove the corollary. What happens if the boundary condition is not homogeneous?

Exercise 16. State and prove a similar result to Corollary 1.73 assuming that Ω is a real interval $[a,b]$ and the boundary conditions instead are homogeneous Neumann conditions; i.e., $u_x(a) = u_x(b) = 0$. (Here we are assuming there is one spatial variable.)

Exercise 17. Repeat Exercise 16 but using *periodic boundary* conditions i.e., $u(a) = u(b) = r$. (Here we are assuming there is one spatial variable.)

6. Some Other Interesting PDE's & Closing Remarks

Remark 1.74. The list of PDE's given in this chapter is by no means complete or comprehensive. In particular, there are many other interesting partial differential equations, many of which we will not cover in this course. Nonetheless, it is worth knowing the exist.

Example 1.75 (Convection-Diffusion)**.** Let $\mathbf{v}(\mathbf{x},t)$ be a velocity field. An example of a *convection diffusion* equation is:

$$\frac{\partial u}{\partial t} = k\nabla^2 u - \nabla \cdot (\mathbf{v}(x,t)u(\mathbf{x},t)) + R(x,t). \tag{1.30}$$

Here $R(x,t)$ may represent sources or sinks of temperature (or a chemical if $u(\mathbf{x},t)$ is a chemical concentration). This assumes a constant diffusion constant k, which is not always a valid assumption and makes the equation more complex. This models the dispersal of a scalar field (temperature, concentration etc.) assuming there is a prevailing movement (the velocity field). Equations like this (but more complex) can be used to model plumes (in a chemical spill) or the trajectory of particles in a sneeze or cough (of great importance in 2020).

Example 1.76 (Korteweg–de Vries (KdV) equation)**.** The *Korteweg–de Vries (KdV) equation* is a nonlinear third order PDE describing shallow water waves:

$$u_t + u_{xxx} - 6uu_x = 0. \tag{1.31}$$

This equation is actively studied in a number of areas of applied mathematics and physics.

Example 1.77 (Turing Patterns)**.** A non-linear reaction diffusion equation may have the form:

$$\mathbf{u}_t = \mathbf{R}(\mathbf{u}) + \mathbf{K}\nabla^2\mathbf{u}, \tag{1.32}$$

which is a system of PDE's (rather than a single PDE). The vector-valued function $\mathbf{R}(\mathbf{u})$ is usually non-linear. These equations are almost never solvable in closed form, but can produce interesting patterns which illustrate the emergence of order from disorder. Examples of this are called Turing Patterns (after Allan Turing who discovered them without the aid of a computer). Examples of Turing pattern solutions to specific reaction diffusion equations are shown in Fig. 1.13.

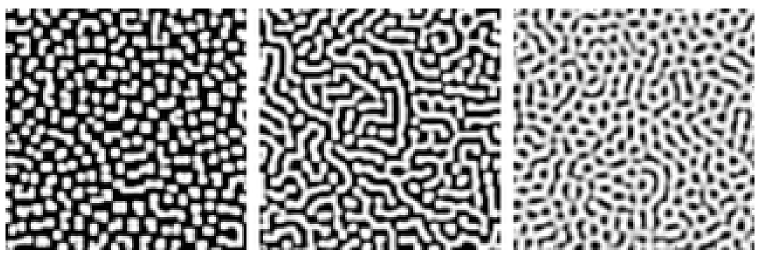

FIGURE 1.13. Reaction diffusion equations exhibit long-term behavior that may produce stripes or spots, like the ones on animals or shells. (Image from Wikipedia https://en.m.wikipedia.org/wiki/Turing_pattern.

Remark 1.78. The remainder of this course is going to be about solving PDE's, which is generally very difficult to do. For the most part, we're going to focus on *well posed* PDE's, which generally means [**Eva15**]:

(1) The problem has a unique solution.
(2) The solution depends continuously on the data (initial and boundary conditions).

The last requirement is largely driven by physics, which respects the fact that nature tends to dislike discontinuities. For us, "solve" almost always is going to mean "write down a formula." For general PDE's, this is not possible, so in a sense, the equations we study are instructive exceptions.

Remark 1.79. Evans [**Eva15**] provides some rules of thumb to carry with you about PDE's.

(1) Nonlinear PDE's are harder than linear PDE's. In general a Linear PDE can be solved by some technique we'll learn in this class.
(2) Higher order PDE's are worse than lower order PDE's.
(3) A system of PDE's is harder to solve than a single PDE.
(4) PDE's with many variables are harder to solve than PDE's with two or three variables.

CHAPTER 2

The Heat Equation and Separation of Variables

The goals of this chapter are to derive the heat equation and to illustrate the method of separation of variables for deriving a generic solution to the heat equation without considering a specific initial condition. Those of you who took Math 251 will already remember how to find the coefficients of the Fourier series we will derive, but please don't give it away. We'll discuss initial conditions and Fourier series in detail in the next chapter.

1. Two Derivations of the Heat Equation

Remark 2.1. In this section, we'll derive the heat equation twice: once in dimension one and then again in dimension three. The derivation is (basically) the same in each dimension, the version in three dimensions explicitly uses the Divergence Theorem.

Derivation 2.2 (Heat Equation in One Dimension)**.** The first derivation is a variation of the one in [**Hab03**]. Consider a one-dimensional line segment ("rod") and suppose that $e(x,t)$ is the linear (thermal) energy density, which is related to temperature in a way to be defined shortly. For completeness, $e(x,t)$ is measured in Joules per meter in SI. For an extremely short interval $(x, x+\epsilon)$, we may assume the net energy is given by:

$$E(x,t) = e(x,t)\,\epsilon. \tag{2.1}$$

This implies that:

$$\frac{\partial E}{\partial t} = \frac{\partial e}{\partial t}\epsilon.$$

The total energy in this segment can be attributed to:

(1) thermal energy flux measured in Joules per second, denoted $\varphi(x,t)$ and
(2) any internal energy generation per unit length per time (e.g., caused by a persistent fire), denoted $Q(x,t)$ and measured in Joules per meter per second.

If we assume a sign convention so that flux is positive if energy moves left-to-right, then at time t, the total energy change is given by:

$$\frac{\partial e}{\partial t}\epsilon \approx \underbrace{\varphi(x,t)}_{\text{Flow In}} - \underbrace{\varphi(x+\epsilon,t)}_{\text{Flow Out}} + \underbrace{Q(x,t)\epsilon}_{\text{Generated Energy}}.$$

This is illustrated in Fig. 2.1. Dividing through by ϵ we have:

$$\frac{\partial e}{\partial t} \approx \frac{\varphi(x,t) - \varphi(x+\epsilon,t)}{\epsilon} + Q(x,t).$$

As ϵ goes to zero, this approximation becomes exact. On the right-hand-side we see (the negative of) a Newton quotient. Thus:

$$\frac{\partial e}{\partial t} = \lim_{\epsilon\to 0}\frac{\varphi(x,t) - \varphi(x+\epsilon,t)}{\epsilon} + Q(x,t) = -\frac{\partial \varphi}{\partial x} + Q(x,t).$$

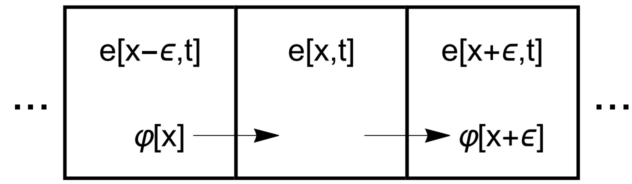

FIGURE 2.1. Flow $\varphi(x)$ is illustrated moving left to right, but that provides a sign convention. The amount of flow into and out of position x must be equal to the total heat density within the region.

Experimentation shows that the relationship between temperature and thermal energy in an object is given by:

$$u(x,t) = c(x)\rho(x)e(x,t), \tag{2.2}$$

where $c(x)$ is the specific heat of the material and $\rho(x)$ is the (linear) density of the material. If we assume these are constants c_0 and ρ_0 for our line segment, we have:

$$\frac{\partial u}{\partial t} = -\frac{1}{c_0\rho_0}\frac{\partial \varphi}{\partial x} + q(x,t), \tag{2.3}$$

where:

$$q(x,t) = \frac{1}{c_0\rho_0}Q(x,t).$$

Unfortunately, this does not relate energy flux to temperature; i.e., we need to relate $u(x,t)$ to $\varphi(x,t)$. This can be accomplished (again somewhat empirically) using Fick's Law or Fourier's Law of Heat, which asserts:

$$\varphi = -K\frac{\partial u}{\partial x}. \tag{2.4}$$

It is worth noting this is also Ohm's law in electrical conductance and Fick's law applies to chemical concentrations and flux instead of heat. Using Eq. (2.4) in Eq. (2.3) yields:

$$\frac{\partial u}{\partial t} = \frac{K}{c_0\rho_0}\frac{\partial^2 u}{\partial x^2} + q(x,t). \tag{2.5}$$

Setting:

$$k = \frac{K}{c_0\rho_0} \tag{2.6}$$

gives the non-homogeneous heat equation in one dimension:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t). \tag{2.7}$$

Remark 2.3. It should be clear when there are no heat sources or sinks then $q(x,t) = 0$ and we have the heat equation from Chapter 1.

Remark 2.4. The one dimensional heat equation is slightly contrived. There are no one dimensional "rods" that can be easily measured. We can, of course, approximate a very long thin cylindrical rod as a one dimensional rod, but we would need to make sure it was insulated everywhere except at the two ends. Otherwise we would need to model radiant heat loss at the surface of the cylinder. Therefore, it's much more realistic to have a three dimensional heat equation, which we'll see is exactly the three dimensional analog of the heat equation we just derived.

THEOREM 2.5 (Divergence Theorem). *Let* $\mathbf{F}$ *be a* C^1 *vector field (continuously differentiable) and suppose* $\Omega \subset \mathbb{R}^3$ *is a three dimensional region with a piecewise smooth boundary* $S = \partial\Omega$. *Then:*

$$\int_{\Omega} \nabla \cdot \mathbf{F}\, d\Omega = \int_{S} \mathbf{F} \cdot \mathbf{n}\, dS, \tag{2.8}$$

where $\mathbf{n}$ *is a unit normal vector field to the surface at each point over the integration.* □

Remark 2.6. The integrals in Eq. (2.8) are triple and double integrals respectively. The fact that we are integrating over volume on the left and area on the right is clear from the context, so you don't need multiple integral signs. How this notation will vary from author to author.

Remark 2.7. The assumption that $\mathbf{F}$ is C^1 everywhere can be relaxed to C^1 on Ω and just C^0 on $\partial\Omega$, but we will have no reason to do this. Also, it's worth noting that the divergence theorem is coordinate independent; i.e., it's true irrespective of the coordinate system chosen.

Remark 2.8. We will need the following lemma, whose formal proof is non-trivial. We provide a simple sketch that could be formalized into a complete proof.

Lemma 2.9. *Let* $\Omega \subset \mathbb{R}^3$ *be a three dimensional region and supposed that* $F(\mathbf{x})$ *is a* C^0 *function defined on* Ω. *If for every set* $B \subset \Omega$ *we have:*

$$\int_B F\, dB = 0,$$

then $F \equiv 0$.

SKETCH OF PROOF. The intuition behind the proof is to suppose that $F \neq 0$. Then for some non-trivial region $B \subset \Omega$, we can assert that $F(\mathbf{x})$ has a single sign for every $\mathbf{x} \in B$. Then for that B,

$$\int_B F\, dB \neq 0,$$

contradicting our assumption. □

Remark 2.10. Incidentally, you can use a slight variation of this lemma to show that the curl of the electrostatic field vanishes [**PM13**].

Derivation 2.11 (Heat Equation in Three Dimension)**.** This second derivation is a variation on the one in [**Log14**]. Suppose we have a region Ω in $\mathbb{R}^3$ with a thermal energy function $e(\mathbf{x}, t)$, where $\mathbf{x} = (x_1, x_2, x_3)$. If we take a small region $B \subset \Omega$, then using the straightforward generalization of Eqs. (2.1) and (2.2) we have the total energy in B is:

$$E = \int_B c\rho u\, dB \tag{2.9}$$

Defining $\boldsymbol{\varphi}(\mathbf{x}, t)$ as the flux vector field *out* of B along the normal $\mathbf{n}$ at the surface $S = \partial B$ is, we have the rate of change of energy in B is:

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\int_B c\rho u\, dB = \underbrace{-\int_{\partial B} \boldsymbol{\varphi} \cdot \mathbf{n}\, dS}_{\text{Energy in/out}} + \underbrace{\int_B q\, dB}_{\text{Energy Generated}}. \tag{2.10}$$

The negative sign on the flux integral arises because we are measuring flow *out*, so if the integral is positive, the energy change should be negative. Applying the divergence theorem and exchanging the integral and derivative yields:

$$\int_B \frac{\partial}{\partial t} c\rho u \, dB = -\int_B \nabla \cdot \varphi \, dB + \int_B q \, dB. \tag{2.11}$$

For simplicity, assume $c = c_0$ and $\rho = \rho_0$ are constant, then rearranging terms we have:

$$\int_B c_0\rho_0 \frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{\varphi} - q \, dB = 0. \tag{2.12}$$

Apply Lemma 2.9 to see:

$$c_0\rho_0 \frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{\varphi} - q = 0.$$

Finally Fick's law (Fourier's Law of Heat Conduction) in three dimensions is:

$$\varphi = -K\nabla u. \tag{2.13}$$

Assuming K is constant and making this substitution we have:

$$c_0\rho_0 \frac{\partial u}{\partial t} - K\nabla \cdot \nabla u - q = 0,$$

which can be rewritten as:

$$\frac{\partial u}{\partial t} = \nabla^2 u + q(\mathbf{x}, t), \tag{2.14}$$

where k is again given by Eq. (2.6). When $q \equiv 0$, we recover the three dimensional homogeneous heat equation.

Remark 2.12. It should be clear, we can generalize this derivation in any number of dimensions and obtain the same equation.

2. Three Lemma's on Boundary Value Problems

Remark 2.13. Our objective in this section is to state three lemma's that will be needed for separation of variables.

Remark 2.14. We state our differential equations in terms of spatial variable x with unknown function $w(x)$. However it would be just as easy to use $w(t)$ and this would not affect the results. Additionally, we use the notation $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ to denote the set of positive integers and $\mathbb{Z}^+_0 = \{0, 1, 2, \dots\}$ to denote the set of non-negative integers.

Lemma 2.15. *Consider the ordinary second order differential equation with boundary conditions:*

$$\begin{cases} w'' + \lambda w = 0 \\ w(0) = w(L) = 0, \end{cases} \tag{2.15}$$

where $L > 0$. This equation has non-trivial real solutions:

$$w_n(x) = b_n \sin\left(\frac{n\pi}{L} x\right).$$

with $b_n \in \mathbb{R}$ only if $\lambda > 0$ and $\lambda = (n\pi/L)^2$ for $n \in \mathbb{Z}^+$.

Proof. We divide the proof into three cases.

Case 1: $\lambda = 0$:
In this case, the differential equation becomes:

$$w'' = 0.$$

This can be solved by direct integration (twice) to obtain:

$$w(x) = C_1 x + C_2.$$

If $w(0) = 0$, then $C_2 = 0$. If $w(L) = 0$, then $C_1 = 0$. Thus $w(x) = 0$ is the trivial solution.

Case 2: $\lambda < 0$:
The characteristic polynomial of the ordinary differential equation is:

$$s^2 + \lambda = 0. \tag{2.16}$$

Therefore, the solutions all have form:

$$w(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}. \tag{2.17}$$

For simplicity, assume $\lambda = -\sigma^2$. Then Eq. (2.17) becomes:

$$w(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x}. \tag{2.18}$$

Using the boundary condition $w(0) = 0$ and substituting we see that:

$$C_1 + C_2 = 0. \tag{2.19}$$

Therefore we can rewrite Eq. (2.18) as:

$$w(x) = \frac{C}{2}\left(e^{\sigma x} - e^{-\sigma x}\right) = C\sinh(\sigma x), \tag{2.20}$$

for some constant C. Here, sinh is the hyperbolic sine function defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

This function has a single root at $x = 0$; i.e., $\sinh(0) = 0$. Therefore, since $L > 0$ it follows that:

$$C\sinh(L) = 0 \implies C = 0.$$

Thus, $w(x) = 0$ is the trivial solution.

Case 2: $\lambda > 0$:
Reasoning as before, Eq. (2.17) still holds, but now assume that $\lambda = \sigma^2$. Then the solution becomes:

$$w(x) = C_1\sin(\sigma x) + C_2\cos(\sigma x). \tag{2.21}$$

With the requirement that $w(0) = 0$, we see that $C_2 = 0$ since $\cos(0) = 1$. Thus, the solution has form:

$$w(x) = C_1\sin(\sigma x). \tag{2.22}$$

If $w(L) = 0$, we must have $\sin(\sigma L) = 0$. This can only happen if:

$$\sigma L = n\pi,$$

for $n \in \mathbb{Z}^+$ because $\sin(n\pi) = 0$. (We can ignore the negative integers, since we can use C_1 to adjust the sign.) Therefore, this equation has non-trivial solution only when:

$$\sigma = \frac{n\pi}{L},$$

which implies:

$$\lambda = \left(\frac{n\pi}{L}\right)^2. \tag{2.23}$$

The result is a non-trivial family of solutions:

$$w_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right),$$

where b_n are arbitrary constants to be determined later. This completes the proof. □

Exercise 18. The hyperbolic cosine function is defined as:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Show that when $\lambda < 0$ the general solution to the ordinary differential equation:

$$w'' + \lambda w = 0$$

is:

$$w(x) = A_1 \cosh[\sigma(x-k)] + A_2 \sinh[\sigma(x-k)],$$

with $\lambda = -\sigma^2$ and constants A_1, A_2 and k. [Hint: Use Eq. (2.18) and relate A_1, A_2, C_1 and C_2. Or you can just take derivatives and show this is the case.]

Definition 2.16 (Eigenfunction/Eigenvalue)**.** Let L be a linear operator a non-zero function u and a scalar λ are an eigenfunction/eigenvalue pair if:

$$Lu = \lambda u. \tag{2.24}$$

The function u may be subject to boundary conditions specified separately.

Remark 2.17. Thus we see that with the boundary conditions $w(0) = w(L)$ and linear operator:

$$L = \frac{d^2}{dx^2}$$

the eigenvalues and eigenfunctions of L are:

$$\lambda = -\left(\frac{n\pi}{L}\right)^2$$

with eigenfunctions:

$$w_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right).$$

This can be verified by showing that:

$$\frac{d^2 w}{dx^2} = \lambda w.$$

In understanding this, it's important to not get confused by the signs. Here, λ is *negative* to make Eq. (2.24) true. Whereas in Eq. (2.15) from Lemma 2.15 the differential equation under consideration is:

$$w'' = -\lambda w.$$

That is, the negative is already taken into consideration. To eliminate the sign confusion, many texts simply refer to this as the eignvalue problem.

Lemma 2.18. *Consider the ordinary second order differential equation with boundary conditions:*

$$\begin{cases} w'' + \lambda w = 0 \\ w'(0) = w'(L) = 0, \end{cases} \tag{2.25}$$

where $L > 0$. This equation has non-trivial real solutions

$$w_n(x) = a_n \cos\left(\frac{n\pi}{L}x\right) \tag{2.26}$$

with $a_n \in \mathbb{R}$ only if $\lambda > 0$ and $\lambda = (n\pi/L)^2$ for $n \in \mathbb{Z}^+$ or $\lambda = 0$ and $w_0(x) = a_0$.

Remark 2.19. The proof of this is almost a direct copy of the previous proof. Given that, we will skip the case when $\lambda < 0$ and leave it as an exercise.

Proof. Assume $\lambda > 0$. It follows from the proof of Lemma 2.15 that Eq. (2.21) sill holds and:

$$w(x) = C_1 \sin(\sigma x) + C_2 \cos(\sigma x),$$

where $\lambda = \sigma^2$. Computation shows:

$$w'(t) = \sigma C_1 \cos(\sigma x) - \sigma C_2 \sin(\sigma x). \tag{2.27}$$

If $w'(0) = 0$, then $C_1 = 0$. If $w'(L) = 0$, then by a similar argument as the one in the proof of Lemma 2.15 we see that if:

$$\sigma = \frac{n\pi}{L},$$

with $n \in \mathbb{Z}^+$, then:

$$-\sigma C_2 \sin(\sigma x) = 0.$$

Thus when $\lambda > 0$, the non-trivial family of solutions to the boundary value problem is:

$$w_n(x) = a_n \cos\left(\frac{n\pi}{L}x\right).$$

Now assume $\lambda = 0$. Then if $w(x) = a_0$ for some $a_0 \in \mathbb{R}$, we see that $w'(x) = 0$ for all x, satisfying the boundary conditions. Substitution shows this solution satisfies the ODE. □

Exercise 19. Complete the missing case in the proof of Lemma 2.21.

Remark 2.20. From the above proof, we can see that the eigenvalues for the boundary values problem given in Eq. (2.25) are:

$$\lambda = -\left(\frac{n\pi}{L}\right)^2$$

while the eigenfunctions are given in Eq. (2.26) or the eigenvalue $\lambda = 0$ with the eigenfunctions a_0 for any value a_0.

Exercise 20. Solve the boundary value problem:

$$\begin{cases} w'' + \lambda w = 0 \\ w'(0) = w(L) = 0, \end{cases}$$

where $L > 0$. In your solution highlight, the eigenvalues and eigenfunctions.

Exercise 21. Compute the eigenvalues and eigenfunctions for the linear operator:

$$L = \frac{d^2}{dx^2} + \frac{d}{dx}$$

assuming the boundary conditions $u(0) = u(L) = 0$. Hint: The general solution to the ordinary differential equation $u'' + u' - \lambda u = 0$ is:

$$u(x) = \begin{cases} e^{-\frac{x}{2}}\left[C_1 \cos\left(\frac{\sqrt{-4\lambda-1}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{-4\lambda-1}}{2}x\right)\right] & \text{if } \lambda < -\frac{1}{4} \\ e^{-\frac{x}{2}}\left[C_1 \exp\left(\frac{\sqrt{1+4\lambda}}{2}x\right) + C_2 \exp\left(-\frac{\sqrt{1+4\lambda}}{2}x\right)\right] & \text{if } \lambda > -\frac{1}{4} \\ C_1 \exp\left(-\frac{x}{2}\right) + C_2 x \exp\left(-\frac{x}{2}\right) & \text{if } \lambda = -\frac{1}{4}. \end{cases}$$

Lemma 2.21. *Consider the ordinary second order differential equation with periodic boundary conditions:*

$$\begin{cases} w'' + \lambda w = 0 \\ w(-L) = w(L) \\ w'(-L) = w'(L) = 0, \end{cases} \tag{2.28}$$

where $L > 0$. This equation has non-trivial real solutions

$$w_n(x) = a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right). \tag{2.29}$$

with $a_n, b_n \in \mathbb{R}$ for $\lambda > 0$ and $\lambda = (n\pi/L)^2$ and constant solution $w_0(x) = a_0$ for $\lambda = 0$.

Proof. Again, we will will skip the case when $\lambda < 0$ and now also the case when $\lambda = 0$ and leave them as an exercise.

If $\lambda = 0$, then the only solution is $C_1 x + C_2$. The boundary conditions imply that $C_1 = 0$ but C_2 can be arbitrary. Therefore we have the constant solution $w_0(x) = a_0$.

Assume $\lambda > 0$. It follows from the proof of Lemma 2.15 that Eq. (2.21) sill holds and:

$$w(x) = C_1 \sin(\sigma x) + C_2 \cos(\sigma x),$$

where $\lambda = \sigma^2$. Computation shows:

$$w'(t) = \sigma C_1 \cos(\sigma x) - \sigma C_2 \sin(\sigma x). \tag{2.30}$$

The first boundary conditions requires:

$$C_1 \sin(\sigma L) + C_2 \cos(\sigma L) = C_1 \sin(-\sigma L) + C_2 \cos(-\sigma L).$$

Since $\cos(L) = \cos(-L)$ and $-\sin(L) = \sin(-L)$ this implies that:

$$2C_1 \sin(\sigma L) = 0.$$

Likewise, the second boundary condition implies

$$\sigma C_1 \cos(\sigma L) - \sigma C_2 \sin(\sigma L) = \sigma C_1 \cos(-\sigma L) - \sigma C_2 \sin(-\sigma L),$$

which implies that:

$$2\sigma C_2 \sin(\sigma L) = 0.$$

As before, if $\sigma = n\pi/L$, then both of these equations can be true simultaneously and C_1 and C_2 can take on any values. Thus, both $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ are eigenfunctions for the eigenvalue $\lambda = \sigma^2 = (n\pi/L)^2$. We conclude that:

$$w_n(x) = a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

This completes the proof. □

Exercise 22. Show that only the trivial solution arises when $\lambda < 0$ in Lemma 2.18, thus complete the proof.

3. Solving the Homogeneous Heat Equation with Separation of Variables

Remark 2.22. The remainder of this chapter will be devoted to solving the heat equation using the separation of variables techniques. We stress, this approach works for the homogeneous heat equation with homogeneous boundary conditions (Dirichlet or Neumann), periodic boundary conditions or Robin boundary conditions (which we have not discussed yet).

Remark 2.23. For the remainder of this chapter, we will omit the initial condition and return to it when we discuss Fourier series in the next chapter.

Derivation 2.24. We consider the one-dimensional heat equation with homogeneous Dirichlet boundary condition:

$$\begin{aligned} &u_t = k u_{xx} \\ &u(0,t) = u(L,t) = 0 \end{aligned}$$

Assume (for the sake of argument) that $u(x,t) = v(t)w(x)$, where $v(t)$ and $w(x)$ are two C^2 univariate functions of time and space respectively. If $u(x,t)$ satisfies the heat equation, then:

$$u_t = v'(t)w(x) = kv(t)w''(x) = ku_{xx}.$$

From this we deduce that:

$$\frac{1}{k}\frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)}.$$

The left-hand-side of this equation is strictly a function of t while the right-hand-side is strictly a function of x. Therefore, if this were to hold for all x and t, it follows that

$$\frac{1}{k}\frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)} = -\lambda,$$

where $\lambda \in \mathbb{R}$ is a constant and we use $-\lambda$ to make it consistent with our lemmas. We now have two ordinary differential equations:

Time ODE:

$$v' = -k\lambda v$$

Space ODE:

$$\begin{aligned} & w'' = -\lambda w \\ & w(0) = w(L) = 0. \end{aligned}$$

Notice we have imposed the boundary conditions on the space ODE because *for all* t, $u(0,t) = u(L,t) = 0$, so unless $v(t) = 0$ (i.e., we have a trivial solution), we'll require $w(0) = w(L) = 0$.

From Lemma 2.15, we know there is a family of solutions to the space ODE given by:

$$w_n(x) = a_n \sin\left(\frac{n\pi}{L}x\right).$$

with:

$$\lambda = \left(\frac{n\pi}{L}\right)^2.$$

We can now solve the time ODE. Recall from Remark 1.11, the solution to the time ODE is:

$$v(t) = A\exp\left(-k\lambda t\right). \tag{2.31}$$

Substituting in the value of λ (as a function of n) we have:

$$v_n(t) = A_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right],$$

where A_n is an arbitrary constant that we are going to absorb into the b_n. Note that the the operator $L = \partial_t - k\partial_{xx}$ is linear. Therefore we can add the individual solutions $u_n(x,t) = v_n(t)w_n(x)$ to obtain a generic solution to the heat equation with homogeneous Dirichlet boundary values:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right] \sin\left(\frac{n\pi}{L}x\right). \tag{2.32}$$

Thus we have proved a proposition.

Proposition 2.25. *Given coefficients b_n ($n \in \mathbb{Z}^+$), if the series:*

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right] \sin\left(\frac{n\pi}{L}x\right). \tag{2.33}$$

may be differentiated term-by-term in both x and t, then $u(x,t)$ solves the one-dimensional heat equation with homogeneous Dirichlet boundary conditions. □

Remark 2.26. We are not going to worry a great deal about whether or not term-by-term differentiation is allowed in the solution $u(x,t)$ defined in Eq. (2.33). Suffice it to say, it is allowed for all the choices of b_n we require.

Remark 2.27. It is worth noting now that we have not determined *how* to identify the coefficients b_n. This will be a major topic of discussion when we introduce initial conditions and discuss Fourier series and Fourier decomposition in the next chapter.

Proposition 2.28. *Given coefficients a_n for $n = 0, 1, \dots$ if the series*

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\cos\left(\frac{n\pi}{L}x\right). \tag{2.34}$$

may be differentiated term-by-term in both x and t, then $u(x,t)$ solves the heat equation with Neumann boundary conditions:

$$\begin{aligned} &u_t = ku_{xx}\\ &u_x(0,t) = u_x(L,t) = 0 \end{aligned}$$

Exercise 23. Use separation of variables to derive Eq. (2.34) in Proposition 2.28. [Hint: Use Lemma Lemma 2.18. Don't forget to include the constant solution corresponding to $\lambda = 0$.]

Exercise 24. Use separation of variables to derive a solution to the following heat equation with only boundary conditions given:

$$\begin{aligned} &u_t = ku_{xx}\\ &u(0,t) = u_x(L,t) = 0. \end{aligned}$$

Explain the physics of this scenario.

4. Heat Equation in a Circle

Remark 2.29. In this section, we will deal with the heat equation on a circle, which really just means introducing periodic boundary conditions:

$$\begin{aligned} u(-L,t) &= u(L,t)\\ u_x(-L,t) &= u_x(L,t). \end{aligned}$$

By enforcing this boundary condition, the behavior of the heat distribution will act as though the geometry of the line has been bent into a circle.

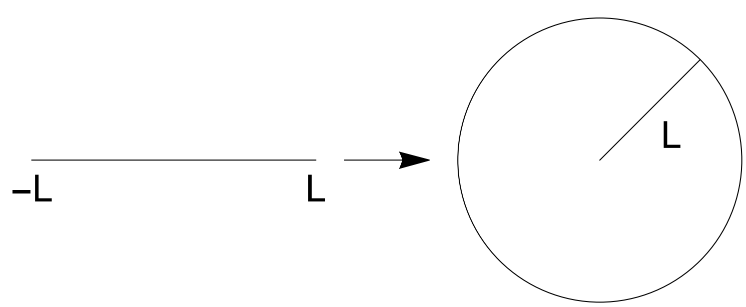

Figure 2.2. Blah

Haberman [Hab03] justifies these boundary conditions by noting that the resulting eigenvalues are identical to those we've already seen (see Lemma 2.21). However, this derivation fails to illustrate the impact geometry can have on the eigenvalues and eigenfunctions of an operator. Consequently, we will state a result with the interval $[0, L]$ "bent" into a circle.

Derivation 2.30. We now consider the one-dimensional heat equation with periodic boundary conditions:

$$\begin{aligned} &u_t = ku_{xx} \\ &u(0,t) = u(L,t) \\ &u_x(0,t) = u_x(L,t) \end{aligned}$$

Following the process of separation of variables as in Derivation 2.24, we obtain two ordinary differential equations:

Time ODE:

$$v' = -k\lambda v$$

Space ODE:

$$\begin{aligned} &w'' = -\lambda w \\ &w(-L) = w(L) \\ &w'(-L) = w'(L). \end{aligned}$$

From Lemma 2.21, we know there is a family of solutions to the space ODE given by:

$$w_n(x) = a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \cos\left(\frac{n\pi}{L}x\right).$$

with:

$$\lambda = \left(\frac{n\pi}{L}\right)^2,$$

for $n = 0, 1, 2, \ldots$. We already have a solution for $v(t)$ from Eq. (2.31) in Derivation 2.24. Substituting our value λ into this expression yields:

$$v_n(t) = A_n \exp\left[-k\lambda t\right] = A_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right].$$

We note the case when $\lambda = 0$ is covered when $n = 0$ and in this case we have a constant $v_0(t) = A_0$ while $w_0(x) = a_0$. Therefore, computing $u_n(x,t) = v_n(t)w_n(x)$ yields the generic solution to the heat equation with periodic boundary conditions on $[0, L]$:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\cos\left(\frac{n\pi}{L}x\right) + b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\sin\left(\frac{n\pi}{L}x\right).$$

5. Asymptotic Properties of the Heat Equation

Proposition 2.31. *In the limit as $t \to \infty$, the general solution for the heat equation with homogeneous Dirichlet, Neumann or periodic boundary conditions approaches a constant function on $[0, L]$, assuming we can pass the limit through the sum*[1].

[1] Doing so requires the series solution to converge uniformly on the interval $[0, L]$ to a "reasonable" function, which it will in every case we care about.

PROOF SKETCH. Without loss of generality, we will use the solution with homogeneous Dirichlet boundary conditions. The remaining cases are left as an exercise.

Recall:

$$w_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right).$$

and

$$v_n(t) = A_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right],$$

and $u_n(x,t) = v_n(t)w_n(x)$. If we assume that:

$$\lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty} \lim_{t\to\infty} u_n(x,t),$$

then we see:

$$\lim_{t\to\infty} v_n(t) = 0$$

and

$$\lim_{t\to\infty} w_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right),$$

which is finite everywhere in $[0, L]$. Therefore:

$$\lim_{t\to\infty} u_n(x,t) = 0,$$

which implies:

$$\lim_{t\to\infty} u(x,t) = 0$$

for $x \in [0, L]$. □

EXERCISE 25. Complete the proof of Proposition 2.31 by showing that it holds for the Neumann and periodic boundary condition cases.

Remark 2.32. When the coefficient a_0 is not zero, then solutions to the heat equation for the boundary conditions given in Proposition 2.31 will approach the constant a_0 rather than 0. The fact that these solutions approach a constant value illustrates the idea behind diffusion. Diffusion smooths out all spatial discontinuities or differences in any initial condition. This is exactly what we'd expect from diffusion.

Example 2.33. To illustrate the previous remark, we find a solution to the heat equation with $k = \frac{1}{10}$, homogeneous Neumann boundary conditions and initial condition $u(x,0) = 1 + \cos(\pi x)$ on the interval $[0,1]$. Notice that $u_x(x,0) = -\pi \sin(\pi x)$, which satisfies the homogeneous Neumann boundary conditions. The initial condition is shown in Fig. 2.3(left).

The generic solution to the heat equation with homogeneous Neumann boundary conditions is:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right] \cos\left(\frac{n\pi}{L}x\right).$$

From this we have:

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) = 1 + \cos(\pi x).$$

We know $L = 1$ and $k = \frac{1}{10}$ and by matching coefficients we conclude that $a_0 = 1$ and $a_1 = 1$ and $a_n = 0$ for $n \geq 2$. Therefore, the exact solution to the initial value problem is:

$$1 + \exp\left(-\frac{\pi^2}{10}t\right)\cos(\pi x)$$

The resulting solution is illustrated Fig. 2.3(right). The mean initial temperature in the rod

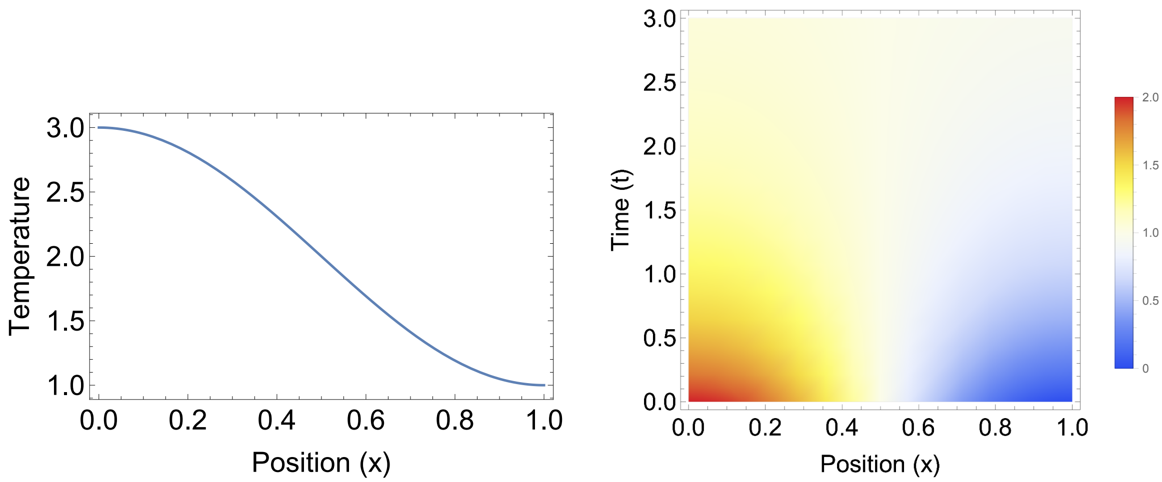

FIGURE 2.3. (Left) The initial temperature distribution of a very thin rod that is perfectly insulated on its ends and that does not radiate heat from anywhere. (Right) The time varying heat distribution of the rod.

is:

$$\bar{u}_0 = \int_0^1 1 + \cos(x)\,dx = x + \frac{1}{\pi}\sin(\pi x)\Big|_0^1 = 1.$$

As we might expect, in the long-run the temperature equilibrates to the mean initial temperature, thus illustrating the diffusion and mixing of hot and cold in the perfectly insulated rod.

Remark 2.34. In the previous example, we could read the coefficients from the initial conditions directly. In the next chapter, we study Fourier series and derive formulas for computing these for given initial conditions.

CHAPTER 3

Fourier Series and Initial Conditions in the Heat Equation

1. Definitions and Preliminaries

Remark 3.1. In this chapter, we will follow historical precedent and define Fourier series on the interval $[-L, L]$. In doing so, we will construct formulae for determining the coefficients a_0, a_n and b_n from the previous chapter. However, this should create some tension in your mind because we spent the last chapter solving problems on $[0, L]$. We can easily translate solutions back and forth or we can find formulae for our coefficients on the interval $[0, L]$. We will illustrate both approaches.

Definition 3.2 (Even and Odd Functions)**.** A function $f(x)$ is *even* if $f(-x) = f(x)$. It is *odd* if $f(-x) = -f(x)$.

Example 3.3. The classic examples of even and odd functions are $f(x) = x^2$ and $f(x) = x$ respectively. Recall from trigonometry that $\cos(x)$ is an even function while $\sin(x)$ is an odd function.

2. Fourier Series

Definition 3.4 (Fourier Series)**.** A *Fourier Series* defined on the interval $[-L, L]$ is an infinite series of the form:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \tag{3.1}$$

Remark 3.5. Much like a Taylor series for a function $f(x)$, the idea is to construct a Fourier series for a function so that:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

This is called the *Fourier decomposition* of the function. We will see, however, that this is not always possible to enforce equality in the strictest sense. Therefore, following [**Olv14, Hab03**], we we will write:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

to indicate that equality may not hold for every value of x.

Lemma 3.6. *Suppose $m, n \in \mathbb{Z}_+$. Then:*

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\, dx = \begin{cases} 0 & \textit{if } m \neq n \\ L & \textit{otherwise.} \end{cases} \tag{3.2}$$

Proof. Assume $n \neq m$. Apply the trigonometric identity:

$$\sin(\theta)\sin(\varphi) = \frac{1}{2}\left[\cos(\theta-\varphi) - \cos(\theta+\varphi)\right].$$

The integral in Eq. (3.2) becomes:

$$\int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{n-m}{L}\pi x\right) - \cos\left(\frac{n+m}{L}\pi x\right)\right] dx = \\ \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{n-m}{L}\pi x\right) - \frac{L}{(n+m)\pi}\sin\left(\frac{n+m}{L}\pi x\right)\right]\Bigg|_{-L}^{L}$$

Using $\sin(-\theta) = -\sin(\theta)$, we can simplify the result as:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2\pi}\left\{2\sin[(n-m)\pi] - 2\sin[(n+m)\pi]\right\}.$$

If $n \neq m$, then $n-m$ and $n+m$ are non-zero integers and thus $\sin[(n-m)\pi] = \sin[(n+m)\pi] = 0$.

Now suppose $n = m$. Apply the trigonometric identity:

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

Then:

$$\int_{-L}^{L} \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right) = \frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Bigg|_{-L}^{L} = L.$$

This completes the proof. □

Lemma 3.7. *Suppose $m, n \in \mathbb{Z}_0^+$. Then:*

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } n = m \neq 0 \\ 2L & \text{if } n = m = 0. \end{cases} \tag{3.3}$$

Exercise 26. Prove Lemma 3.7. You may find the following trigonometric identities useful:

$$\cos(\theta)\cos(\varphi) = \frac{1}{2}(\cos(\theta-\varphi) + \cos(\theta+\varphi))$$
$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$
$$\cos(0) = 1$$

Lemma 3.8. *Suppose $m, n \in \mathbb{Z}_0^+$. Then:*

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = 0 \tag{3.4}$$

PROOF. Apply the identity:

$$\sin(\theta)\cos(\varphi) = \frac{1}{2}\left[\sin(\theta+\varphi) + \sin(\theta-\varphi)\right].$$

The integral becomes:

$$\frac{1}{2}\int_{-L}^{L} \sin\left[\frac{(n+m)\pi x}{L}\right] + \sin\left[\frac{(n-m)\pi x}{L}\right]\,dx = \\ \frac{1}{2}\left\{-\frac{L}{(n+m)\pi}\cos\left[\frac{(n+m)\pi x}{L}\right] - \frac{L}{(n-m)\pi}\cos\left[\frac{(n-m)\pi x}{L}\right]\right\}\Bigg|_{-L}^{L}.$$

Cosine is an even function so $\cos(-x) = \cos(x)$. Therefore, evaluating at $x = L$ and $x = -L$ we see:

$$\cos\left[(n+m)\pi\right] - \cos\left[-(n+m)\pi\right] = 0 = \cos\left[(n-m)\pi\right] - \cos\left[-(n-m)\pi\right].$$

Thus the integral is zero as required. □

Proposition 3.9. *Suppose that* $f : [-L, L] \to \mathbb{R}$ *and*

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

Then:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \tag{3.5}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \tag{3.6}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \tag{3.7}$$

assuming these integrals exist and we may integrate the Fourier series term-by-term.

PROOF. If $m = 0$, then $1 = \cos(m\pi x/L)$. It follows that:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L}\left[a_0\cos\left(\frac{m\pi x}{L}\right) + \right. \\ \left. \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\right]\,dx$$

Passing the integrals through the sums and applying Lemmas 3.7 and 3.8 implies:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} a_0\,dx = 2La_0.$$

Therefore:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

For $m \neq 0$, we compute:

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\,dx = \int_{-L}^{L}\left[a_0\cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\right]\,dx$$

Now suppose that $m = n$. Applying Lemmas 3.7 and 3.8 again implies:

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \int_{-L}^{L} a_n\cos^2\left(\frac{n\pi x}{L}\right)\,dx = a_n L,$$

because (again) $a_0 = a_0\cos(m\pi x/L)$ for $m = 0$. Therefore:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

The equality

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx,$$

holds by a similar argument but by multiplying by $\sin(n\pi x/L)$. □

Remark 3.10. In order for a function to have a Fourier series, these integrals must exist for all coefficients. In particular, this means that $f(x)$ must at least be integrable on $[-L, L]$ (or an appropriate interval in question). Thus, (for example) this eliminates function with asymptotes.

Example 3.11. We will find a Fourier series for the function:

$$f(x) = \frac{x^2}{4}$$

on the interval $[-\pi, \pi]$. Using this interval will remove the π terms in the functions and get us an interesting corollary. First we compute b_n. We can use an integral table or a computer algebra system (like Maple or Mathematica) to compute:

$$\int \frac{x^2}{4}\sin(nx)\,dx = \frac{2x\sin(nx)}{4n^2} - \frac{(n^2x^2-2)\cos(nx)}{4n^3}.$$

Notice the function:

$$g(x) = \frac{(n^2x^2-2)\cos(nx)}{4n^3}$$

is *even* because $\cos(x)$ and $ax^2 - b$ are both even. Therefore evaluating $g(x)$ at $x = \pi$ and $x = -\pi$ yields the same values. At the same time, $\sin(n\pi) = 0$ and therefore we can conclude that:

$$\int_{-\pi}^{\pi} \frac{x^2}{4}\sin(nx)\,dx = \frac{2x\sin(nx)}{4n^2} - \frac{(n^2x^2-2)\cos(nx)}{4n^3} = 0.$$

If you've already used a computer, you could have jumped right to this step. Therefore, $b_n = 0$ for all n. This is not surprising. We know $f(x)$ is an even function so we would not expect it to be the result of adding up odd functions like $\sin(nx)$.

Next we compute a_n for $n \geq 1$. We have:

$$\int \frac{x^2}{4}\cos(nx)\,dx = \frac{(n^2x^2-2)\sin(nx)}{4n^3} + \frac{2x\cos(nx)}{4n^2}.$$

Evaluating at $x = -\pi$ and $x = \pi$ and subtracting yields:

$$\int_{-\pi}^{\pi} \frac{x^2}{4}\cos(nx)\,dx = \frac{4\pi\cos(\pi n)}{4n^2},$$

because $\sin(n\pi) = 0$ and $x\cos(x)$ is odd. This can be simplified by noting that $\cos(\pi n) = (-1)^n$. Therefore we conclude:

$$a_n = \frac{1}{\pi}\left(\frac{(-1)^n\pi}{n^2}\right) = \frac{(-1)^n}{n^2}.$$

for $n \geq 1$. Finally:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{x^2}{4}\,dx = \frac{\pi^2}{12}.$$

We conclude that:

$$\frac{x^2}{4} \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

This is illustrated in Fig. 3.1.

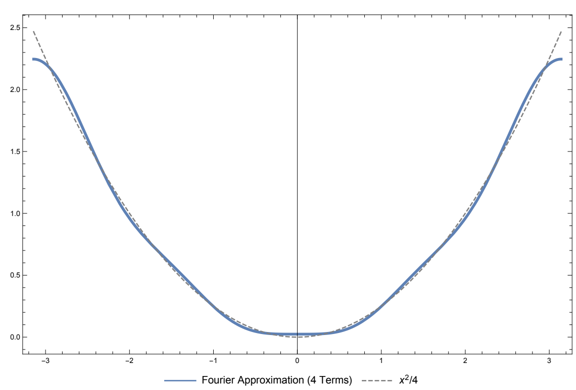

FIGURE 3.1. An illustration of the approximation of $f(x)$ on the interval $[-\pi, \pi]$ using four Fourier terms.

Corollary 3.12. *Assuming the Fourier decomposition for $x^2/4$ converges to the function itself on $[-\pi, \pi]$, then:*

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}. \tag{3.8}$$

PROOF SKETCH. Since we defined the Fourier decomposition on $[-\pi, \pi]$, we have every right to expect that:

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi)$$

by setting $x = \pi$. Note (again) that:

$$\cos(n\pi) = (-1)^n$$

Therefore we have:

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Subtract $\pi^2/12$ from both sides to obtain Eq. (3.8). □

Remark 3.13. This is Basel Problem first solved by Euler in 1734 (non-rigorously first, then rigorously later). It is one of the early hints at the deep connections between Number Theory and Fourier (Harmonic) Analysis. A different proof using Fourier methods can be produced using Pareval's Theorem.

Remark 3.14. Shortly we will prove that the Fourier series for $x^2/4$ does converge and that we are justified in deriving a solution to the Basel problem in this way.

Definition 3.15 (Inner Product on Function Spaces)**.** Let $f, g : [a, b] \to \mathbb{R}$ be two functions defined on the interval $[a, b]$. Then the *inner product* of these functions is:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx,$$

provided it exists. If $f, g : [a, b] \to \mathbb{C}$ then the *(complex) inner product* is:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\,dx,$$

provided it exists. Here $\overline{g(x)}$ is the complex conjugate of $g(x)$.

Definition 3.16 (Orthogonal Functions)**.** Two functions $f, g : [a, b] \to \mathbb{R}$ (respectively $f, g : [a, b] \to \mathbb{C}$) are *orthogonal* if $\langle f, g \rangle = 0$.

Corollary 3.17. *If $n \neq m$ are integers, then $\sin(n\pi x/L)$ and $\sin(m\pi x/L)$ are orthogonal on the interval $[-L, L]$. The same also holds for $\cos(n\pi x/L)$ and $\cos(m\pi x/L)$ for $m \neq n$ and $\sin(m\pi x/L)$ and $\cos(n\pi x/L)$ irrespective of m and n.* □

Remark 3.18 (What is a Fourier Decomposition?)**.** Recall from Vector Calculus that in $\mathbb{R}^2$, any vector $\vec{\mathbf{x}}$ can be written as:

$$\vec{\mathbf{x}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}},$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the standard basis vectors. Using the dot product we note that also that $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ because the vectors are orthogonal. (The dot product of orthogonal vectors is zero.) The coefficients (a, b) are the coordinates of the vector written in the standard basis. The same is true of the Fourier coefficients. We imagine the function $f : [-L, L] \to \mathbb{R}$ as a vector in an infinite dimensional space. The functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ are basis vectors in this space. The Fourier decomposition is just an expression of the function $f(x)$ in this basis. Note:

$$\vec{\mathbf{x}} \cdot \hat{\mathbf{i}} = a$$

$$\vec{\mathbf{x}} \cdot \hat{\mathbf{j}} = b$$

By taking a dot product of the vector $\vec{\mathbf{x}}$ with the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, we recover the coefficients. This is exactly what Eqs. (3.5) to (3.7) are doing – with an appropriate rescale because $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ are not unit vectors, as we showed in Lemmas 3.6 and 3.7.

3. Solving the Heat Equation on $[0, L]$

Remark 3.19. Having established some basic theory on Fourier series, we can now return to the problem of solving the heat equation with an initial condition $u(x, 0) = f(x)$.

Lemma 3.20. *If $f : [0, L] \to \mathbb{R}$ is a function and*

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

then:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

□

Exercise 27. Prove Lemma 3.20 by first showing that

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n. \end{cases}$$

Then argue as we did in Proposition 3.9.

Derivation 3.21. Consider the one-dimensional heat equation with homogeneous Dirichlet boundary conditions and an initial condition:

$$\begin{aligned} &u_t = ku_{xx} \\ &u(0, t) = u(L, t) = 0 \\ &u(x, 0) = f(x). \end{aligned}$$

Recall the solution was given in Eq. (2.32):

$$u(x, t) = \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\sin\left(\frac{n\pi}{L}x\right).$$

except for values of b_n. Note we require:

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

Therefore, we are simply constructing a Fourier decomposition of $f(x)$ using sine functions. We can apply Lemma 3.20 to obtain the values for b_n and we have solved the partial differential equation completely.

Example 3.22. Consider the heat equation with $k = 1$ and $L = 1$ and homogeneous Dirichlet boundary conditions. Suppose:

$$u(x, 0) = \begin{cases} 1 & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Then:

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\int_{\frac{1}{4}}^{\frac{3}{4}} \sin(n\pi x)\,dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right)$$

We can use a sum-to-product identity

$$\cos(\theta) - \cos(\varphi) = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

to compute:

$$\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right) = \frac{-4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(-\frac{n\pi}{4}\right) = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{4}\right),$$

because $\sin(-x) = \sin(x)$. It is perfectly acceptable to stop here (or even the step before) and write:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{4}\right)\sin(n\pi x). \tag{3.9}$$

The approximation for $f(x)$ using 20 terms is shown in Fig. 3.2. Suppose, however, we wish

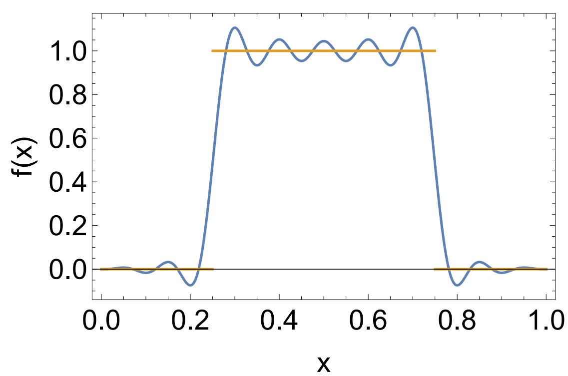

FIGURE 3.2. An approximation of the step function describing the initial heat distribution.

to get rid of those trigonometric terms. We can do it, but it's not obvious from the outside. First, note $n = 2m$ (i.e., even) then $b_n = 0$ because:

$$\sin\left(\frac{n\pi}{2}\right) = \sin(m\pi) = 0.$$

Thus, we will be left with only odd terms. When n is odd, then $n = 2m-1$ (for $m = 1, 2, \dots$) we have:

$$\sin\left(\frac{(2m-1)\pi}{2}\right)\sin\left(\frac{(2m-1)\pi}{4}\right) = (-1)^{f(m)}\frac{\sqrt{2}}{2}.$$

The exact form of $f(m)$ is unclear. Computing a few terms we see a pattern: It turns out

m	1	2	3	4	5	6	7	8
$\sin\left(\frac{(2m-1)\pi}{4}\right)$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

this patten can be described by the function:

$$\sin\left(\frac{(2m-1)\pi}{2}\right)\sin\left(\frac{(2m-1)\pi}{4}\right) = (-1)^{\frac{m(m-1)}{2}}\frac{\sqrt{2}}{2}$$

Therefore, we can write:

$$f(x) \sim \sum_{m=1}^{\infty}(-1)^{\frac{m(m-1)}{2}}\frac{2\sqrt{2}}{(2m-1)\pi}\sin\left((2m-1)\pi x\right).$$

We conclude that:

$$u(x,t) = \sum_{m=1}^{\infty}(-1)^{\frac{m(m-1)}{2}}\frac{2\sqrt{2}}{(2m-1)\pi}\sin\left[(2m-1)\pi x\right]\exp\left[-k\left((2m-1)\pi\right)^2 t\right].$$

Assuming $k = 1/10$, an approximation using 100 terms for various times is shown in Fig. 3.3.

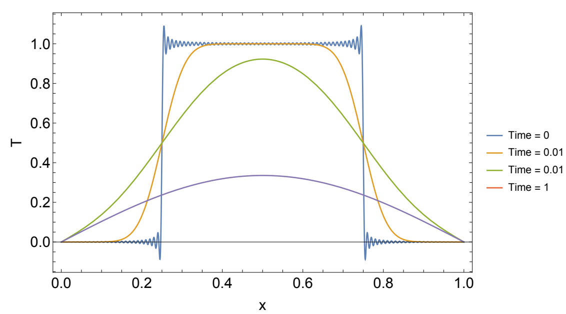

FIGURE 3.3. An approximation of the solution to the heat equation when the initial distribution has a jump discontinuity.

Remark 3.23. It is worth noting that the jump discontinuity could be at the boundary even given the Dirichlet boundary conditions. That is, we could use the initial condition:

$$u(x,0) = \begin{cases} 10 & \text{if } 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

and use the boundary condition $u(0,t) = u(0,L) = 0$.

Remark 3.24. We note also this is an example of a solution that is not really differentiable everywhere at all times and yet we are using it as a solution to a PDE.

Lemma 3.25. *If* $f : [0, L] \to \mathbb{R}$ *is a function and*

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

then:

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx
\end{aligned}$$

□

EXERCISE 28. Prove Lemma 3.25.

Derivation 3.26. Consider the one-dimensional heat equation with homogeneous Neumann boundary conditions and an initial condition:

$$\begin{aligned}
&u_t = ku_{xx} \\
&u_x(0,t) = u_x(L,t) = 0 \\
&u(x,0) = f(x).
\end{aligned}$$

Recall the solution was given in Eq. (2.34):

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right]\cos\left(\frac{n\pi}{L}x\right).$$

except for values of a_0 and a_n. Note we require:

$$u(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right).$$

Therefore, we are simply constructing a Fourier decomposition of $f(x)$ using cosine functions. We can apply Lemma 3.25 to obtain the values for a_0 and a_n and we have solved the partial differential equation completely.

Example 3.27. Consider the same problem data as in Example 3.22 except replace the homogeneous Dirichlet boundary conditions with homogeneous Neumann boundary conditions. Then:

$$a_0 = \int_{\frac{1}{4}}^{\frac{3}{4}} dx = \frac{1}{2}.$$

Likewise:

$$a_n = 2\int_{\frac{1}{4}}^{\frac{3}{4}} \cos(n\pi x)\,dx = \frac{2}{n\pi}\left[\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right].$$

Simplifying further is left to the reader (if so desired). The resulting solution is:

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right]\exp\left[-k\left(n\pi\right)^2 t\right]\cos\left(n\pi x\right)$$

We can illustrate the resulting solution with the density plot in Fig. 3.4.

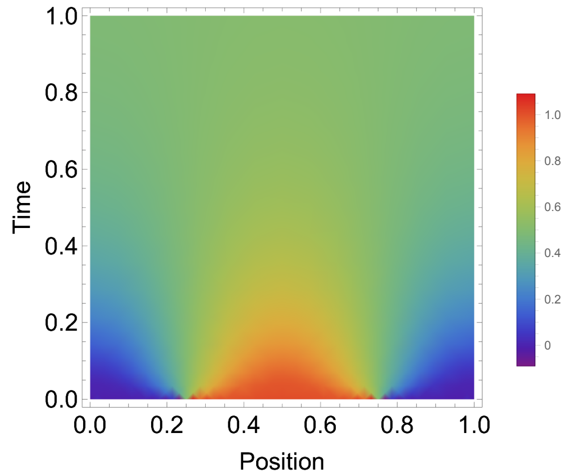

FIGURE 3.4. An illustration of the heat equation with Neumann boundary conditions and a discontinuous initial condition.

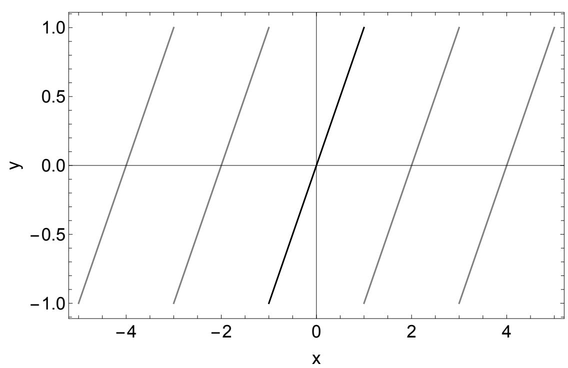

FIGURE 3.5. The periodic extension of the function $f(x) = x$ defined on the interval $[-L, L]$.

4. Convergence of the Fourier Decomposition & Gibbs Phenomena

Remark 3.28. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* with period T if $f(x) = f(x+T)$ for all $x \in \mathbb{R}$. For example, the function $\sin(2\pi f x)$ is periodic with period $1/f$.

Definition 3.29 (Periodic Extension)**.** Suppose $f : [-L, L] \to \mathbb{R}$ is defined. The *periodic extension* of f to $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is the function defined as:

$$\tilde{f}(x) = f(x - 2Lm), \tag{3.10}$$

where m is the unique integer such that $-L \leq (x - 2Lm) \leq L$.

EXERCISE 29. Show that the periodic extension has period $2L$.

Remark 3.30. Definition 3.29 is a complicated way of saying we take f defined on the interval $[-L, L]$ and repeat it over and over in both directions. The same process can be repeated with any interval. For example $[0, L]$, but then we have to figure out what to do when we cross the origin. We illustrate this in Fig. 3.5 with the function $f(x) = x$ defined *only* on the interval $[-1, 1]$.

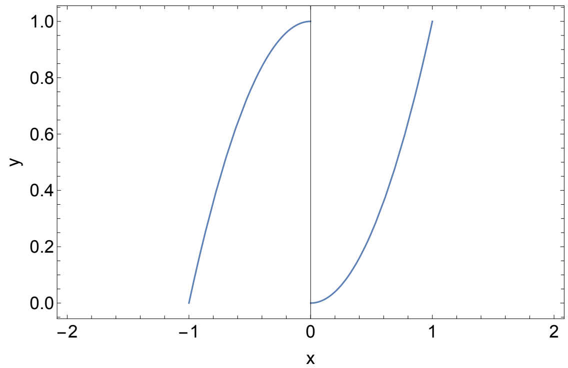

FIGURE 3.6. A piecewise C^1 function on the interval $[-1, 1]$.

Definition 3.31 (Piecewise C^1)**.** A function $f : [-L, L] \to \mathbb{R}$ is piecewise C^1 if the interval can be broken into a finite number of sub-intervals $[l_1, l_2], \ldots, [l_{n-1}, l_n]$ so that:

(1) $f(x)$ is C^1 on (l_i, l_{i+1}) for all i and
(2) $f(x)$ is left (right) continuous at l_{i+1} (l_i).

Remark 3.32. A piecewise C^1 function (sometimes called piecewise smooth, but not always) is a generally well behaved function that has a finite number of jump discontinuities. This eliminates functions with cusps, asymptotes, vertical derivatives or generally anything non-physical. The function:

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -1 \leq x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$$

We illustrate this function in Fig. 3.6, though the initial conditions used in Examples 3.22 and 3.27 are also examples of piecewise C^1 functions.

THEOREM 3.33. *If $f(x)$ is piecewise C^1 on the interval $[-L, L]$ then the Fourier series $\hat{f}(x)$ of $f(x)$ converges to:*

(1) *The periodic extension of $f(x)$ on $\mathbb{R}$ where ever the periodic extension is continuous, or*
(2) *The average:*

$$\hat{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right],$$

if x is a point of discontinuity (i.e., a jump discontinuity). Here:

$$f(x^+) = \lim_{a \to x^+} f(a)$$
$$f(x^-) = \lim_{a \to x^-} f(a)$$

Remark 3.34. You will note, we are being *intentionally vague* about what we mean by *convergence*. In fact we will not even commit to whether that convergence is uniform or pointwise since it turns out it's dependent on the function itself.

Remark 3.35. The *shortest* proof of a form of this theorem is most likely in Rudin [**R+64**]. It's only a paragraph long, but relies on a number of auxiliary results. Haberman [**Hab03**] eschews a proof, focusing on the applications. Olvers [Olv14] has an exceptionally detailed proof. There is also a proof in Asmar [Asm16].

There are a few ways to approach the problem of proving this result (which we will not do). Olver's approach takes a nice detour through Hilbert space. Rudin's approach ignores the possibility of jump discontinuities and deals with the theorem as a part of a discussion on sequences and series of functions. It is a little less general, but a really beautiful proof.

The better part of the 150 years after Fourier lived and died saw mathematicians trying to quantify the conditions under which Fourier series converge and how they converge. Most of modern analysis is built upon and relies on these results. For our purposes, it suffices to think of convergence as being *close enough* for practical physics or engineering (or finance) purposes.

Corollary 3.36 (The Basal Problem)**.** *The equality:*

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

holds.

Proof. The function $x^2/4$ is piecewise C^1 therefore its Fourier series converges to it. Moreover, $\pi^2/4 = -\pi^2/4$ therefore the periodic extension of $x^2/4$ is continuous (though not C^1) on $\mathbb{R}$ therefore there are no points of discontinuity and hence the Fourier series converges to this continuous function. Substituting π into the Fourier series as we did in Corollary 3.12 establishes the result. $\square$

Example 3.37 (Gibb's Phenomena)**.** In Theorem 3.33, we quantify the conditions under which a Fourier series will converge to the function it is approximating, but infinite sums are impossible to compute in practice. Therefore, it is worth understanding what happens when we use a finite sum to approximate a function. We have already seen this in Figs. 3.2 and 3.3. It is tempting to think this is simply a matter of using a short series approximation, but the overshoot seen near the jump discontinuity (approximately a 9% error) persists even as we add terms as shown in Fig. 3.7. This error is called Gibbs phenomena (named for J.

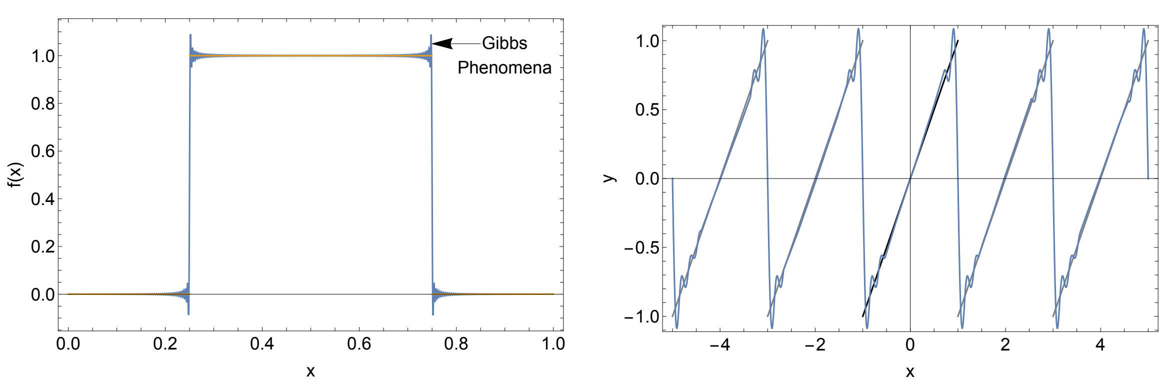

Figure 3.7. (Left) Gibbs Phenomena is persistent even as the number of terms in the Fourier series increases. This over/undershoot is about 9%. (Right) Gibbs phenomena presents as a result of discontinuities in the periodic extension of the function being approximated.

Willard Gibbs, an American physicist). It is present anytime there is a jump discontinuity in the periodic extension of the function being approximated. There are several explanations for the emergence of this phenomena but the most straightforward one is the fact that the Fourier series does *not* converge uniformly to the periodic extension of the underlying function because of the presence of the jump discontinuities.

5. Fourier Sin and Cos Series

Remark 3.38. We have already noticed in computing the Fourier series of $x^2/4$ that $b_n = 0$ for all n. We noted that this is a result of the fact that both $\cos(x)$ and $x^2/4$ are even. We now formalize this observation.

Lemma 3.39. *The sum of two even functions is even. The sum of two odd functions is odd.* □

Exercise 30. Prove the Lemma 3.39.

Lemma 3.40. *The product of two even functions is even. The product of two odd functions is even. The product of an even and odd function is odd.* □

Exercise 31. Prove the Lemma 3.40.

Lemma 3.41. *If $f(x)$ is odd and integrable on $[-L, L]$, then:*

$$\int_{-L}^{L} f(x)\,dx = 0.$$

Proof. We have:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx =$$

$$\int_{0}^{L} f(-x)\,dx + \int_{0}^{L} f(x)\,dx = -\int_{0}^{L} f(x)\,dx + \int_{0}^{L} f(x)\,dx = 0.$$

□

Lemma 3.42. *If $f(x)$ is even and integrable on $[-L, L]$, then:*

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

Exercise 32. Prove Lemma 3.42.

Remark 3.43. The following proposition follows from Lemmas 3.40 to 3.42 and Lemmas 3.20 and 3.25.

Proposition 3.44. *If $f(x)$ is even, then $b_n = 0$ for all n in its Fourier decomposition on $[-L, L]$ and $f(x)$ is represented by a Fourier cosine series whose coefficients may be determined by Lemma 3.25. If $f(x)$ is odd, then $a_0 = a_n = 0$ for all n in its Fourier decomposition on $[-L, L]$ and $f(x)$ is represented by a Fourier sine series whose coefficients may be determined by Lemma 3.20.* □

Remark 3.45. What this means is that when we compute a Fourier sine series of a function on $[0, L]$, we will always recover its periodic extension as an *odd* function. When we compute a Fourier cosine series of a function on $[0, L]$ we will recover its even extension.

Example 3.46. Consider the odd function $f(x) = x$ and compute its Fourier cosine series on $[0, \pi]$ (here $L = \pi$). We have:

$$a_0 = \frac{1}{\pi}\int_0^\pi x\, dx = \frac{\pi}{2}$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)\, dx = \frac{2\left(-1+\cos(n\pi)\right)}{\pi n^2} = \frac{2}{\pi n^2}\left(-1+(-1)^n\right) = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

On $[0, L]$ (and only on this interval) we have:

$$x \sim \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4\cos[(2n-1)x]}{\pi(2n-1)^2}, \tag{3.11}$$

which perfectly reproduces $f(x) = x$, see Fig. 3.8 (left). However, if we look at the plot of this approximation outside the region we see that the Fourier cosine series actually produces an even sawtooth wave, which models $f(x) = |x|$ on $[-\pi, \pi]$. See Fig. 3.8(right). This is a result of using a Fourier cosine series on an odd function.

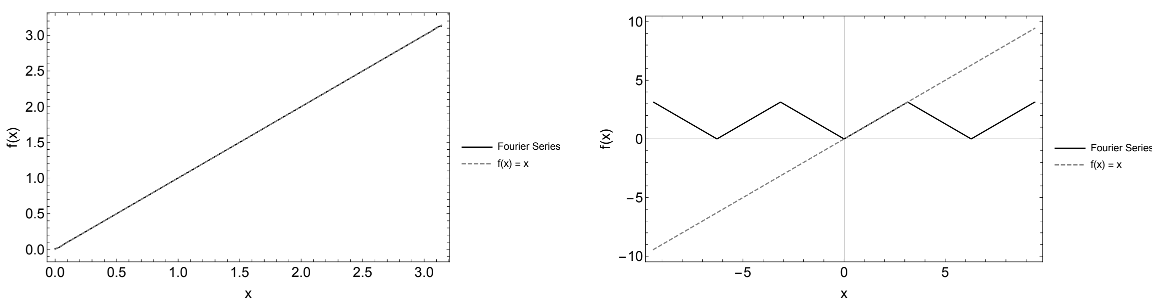

FIGURE 3.8. The Fourier cosine series of an odd function produces an even periodic extension, as expected.

Remark 3.47. We can obtain another remarkable series representation from the previous example. Set $x = 0$ in the Fourier cosine expansion in Eq. (3.11) to obtain:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Simplifying a bit yields:

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots .$$

EXERCISE 33. Compute the Fourier sine series of $f(x) = x^2$ on $[0, 1]$. What does the graph of the periodic extension look like?

Remark 3.48. In *many* books, when dealing with a Fourier cosine series, by convention, the series is written:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

If this is the case, then you can use the same formula for a_0 as a_n because you will be dividing by 2. (See Lemma 3.25.)

6. Validity of Term-by-Term Operations

Remark 3.49. So far, we have been indiscriminately assuming that we can differentiate or integrate Fourier series with no problems. As we will see, this is not necessarily the case. However, we can state some theorems that ensure term-by-term operations are acceptable.

THEOREM 3.50. *If $f(x)$ has a piecewise C^2 and continuous periodic extension, then the Fourier series can be differentiated term-by-term to produce a series representation of $f'(x)$.* □

Example 3.51. Consider the Fourier expansion of $f(x) = |x|$, which we found by computing a cosine series of $f(x) = x$ as illustrated in Fig. 3.8. This function is piecewise C^2 and continuous in its periodic extension (as we see in the figure). Therefore it should be acceptable to differentiate this series to obtain a series approximation for

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Differentiating we obtain:

$$f'(x) \sim \sum_{n=1}^{\infty} \frac{4\sin[(2n-1)x]}{(2n-1)\pi}.$$

We can plot this to see that the series derivative is a reasonable representation of the true derivative (see Fig. 3.9).

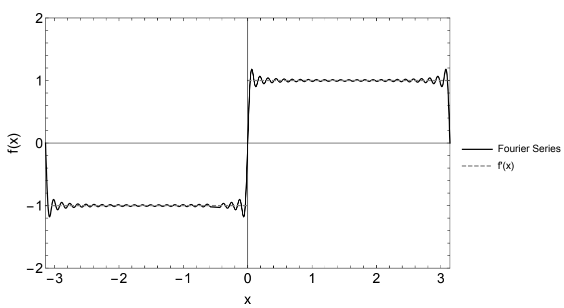

FIGURE 3.9. The series derivative for $|x|$ approximates the periodic extension of the derivative except at jump discontinuities (as expected).

Remark 3.52. As Olver [Olv14] notes there a derivative of a series (in some sense) makes things worse, which is why we require the original function to be both C^2 (ensuring its derivative is C^1, which is required by Fourier's theorem) *and* that the periodic extension is continuous.

Example 3.53. Consider $f(x) = x^3$. This is an odd function and on $[-1, 1]$ it has a Fourier series given by:

$$x^3 = \sum_{n=1}^{\infty} \frac{2(-1)^n \left(n^2\pi^2 - 6\right)}{n^3\pi^3} \sin(n\pi x).$$

However, since x^3 is odd we know it will *not* have a continuous periodic extension. Consequently, we cannot expect the derivative of its Fourier series to faithfully represent the derivative of the function. This is illustrated in Fig. 3.10 (left). We can even compare the

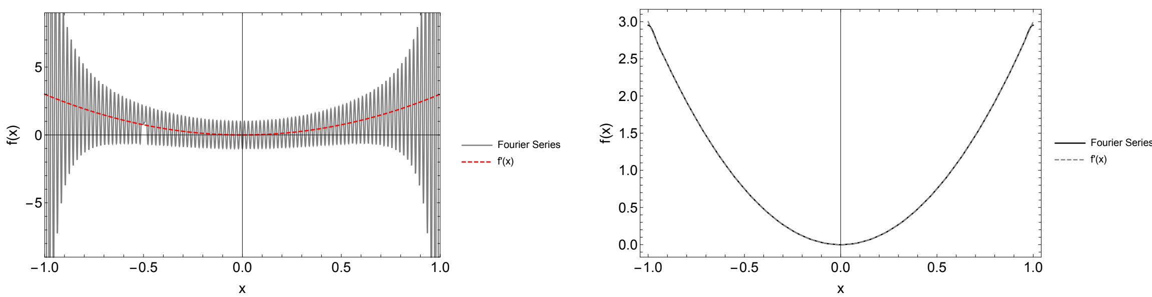

FIGURE 3.10. (Left)The series derivative for x^3 does not approximate the periodic extension of the derivative because x^3 does not have a continuous periodic extension. (Right) The correct Fourier decomposition of $3x^2$ provides a good approximation of the function for even short series.

derivative of the series with the Fourier approximation of $3x^2$. The derivative of the series is:

$$\sum_{n=1}^{\infty} \frac{2(-1)^n \left(n^2\pi^2 - 6\right)}{n^2\pi^2} \cos(n\pi x).$$

Solving for the Fourier series decomposition of $f(x) = 3x^2$ on $[-1, 1]$ yields:

$$3x^2 = 1 + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^2\pi^2} \cos(n\pi x).$$

Remark 3.54. Haberman [**Hab03**] states a few more specific conditions for Fourier sine and cosine series to be differentiable. We'll focus on two in particular.

Proposition 3.55. *A Fourier sine series of $f(x)$ on $[0, L]$ is term-by-term differentiable only if $f(x)$ is piecewise C^2 and $f(0) = f(L) = 0$.*

PROOF SKETCH. The fact that $f(0) = f(L) = 0$ implies that the periodic extension of $f(x)$ is continuous. □

Proposition 3.56. *A Fourier cosine series of $f(x)$ on $[0, L]$ is term-by-term differentiable if $f(x)$ is piecewise C^2.*

PROOF SKETCH. The fact that $f(x) = f(-x)$ for all x implies that the periodic extension of $f(x)$ is continuous. □

Remark 3.57. The following theorem is the last one we need to connect all of these results together with our work on the Heat equation.

THEOREM 3.58. *Suppose $u(x,t)$ is a C^2 function and is given by the eigenfunction expansion (Fourier series with time-varying coefficients) on the interval $[0, L]$*

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{L}\right). \tag{3.12}$$

and $u(x,0) = f(x)$ and satisfies homogeneous Dirichlet or Neumann boundary conditions. The series representation maybe differentiated term-by-term in both x and t. □

Corollary 3.59. *For the Dirichlet conditions $u(0,t) = u(L,t) = 0$ the solution:*

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-k\left(\frac{n\pi}{L}\right)^2 t\right] \sin\left(\frac{n\pi}{L}x\right).$$

solves the heat equation. □

Remark 3.60. The proof of the corollary uses term-by-term differentiation and direct comparison. A similar corollary can be proved for the Neumann boundary conditions.

Remark 3.61. We conclude by observing we can obtain an even stronger result on solutions to the heat equation than the one in Remark 2.32.

THEOREM 3.62. *If $u(x,t)$ is a solution to the heat equation with piecewise continuous initial data $f(x) = u(x,0)$ (or more generally $f(x)$ is absolutely integrable on $[0, L]$) then for any $t > 0$, the solution $u(x,t)$ is infinitely differentiable in x.* □

Remark 3.63. Here we can relax the differentiability requirements a bit because we're assuming continuity. This infinite differentiability is precisely the smoothing discussed in Remark 2.32.

CHAPTER 4

The Wave Equation

1. Deriving The One Dimensional Wave Equation

Definition 4.1 (Newton's Second Law of Motion)**.** If an object of mass m is acted upon by a (vectorial) force $\mathbf{F}$, then the resulting (vectorial) acceleration $\mathbf{a}$ is given by:

$$\mathbf{a} = \frac{\mathbf{F}}{m},$$

or more familiarly: $\mathbf{F} = m\mathbf{a}$. This is *Newton's Second Law of Motion.*

Derivation 4.2. Consider an idealized one dimensional string. Let $u(x,t)$ denote the vertical displacement of the string at time t and at position x. At any given time, we assume there is a tangential force (tension) on the string, which is shown in Fig. 4.1. Assume the string

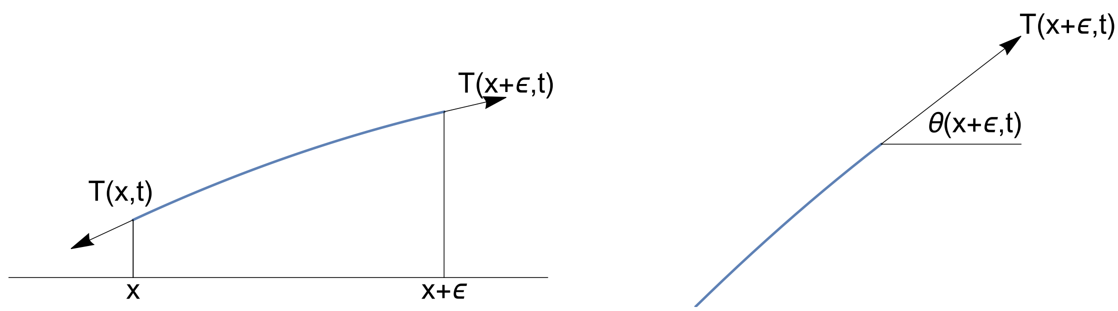

FIGURE 4.1. (Left) The tension is a force tangential to the string at any point along it. (Right) The angle with respect to the horizontal is used to decompose tension into horizontal and vertical components.

as a mass density $\rho(x)$. Then the total mass between point x and $x+\epsilon$ is $\epsilon\rho(x)$ (because the length of the interval is ϵ). If we consider only acceleration in the vertical direction (i.e., we assume horizontal motion is negligible) then from Newton's Second Law we conclude:

$$F_{\text{vert}} = \epsilon\rho(x)\frac{\partial^2 u}{\partial t^2}.$$

Here the acceleration is the second partial derivative of the vertical displacement with respect to time. We must now compute the vertical force F_{vert}. The net (vector) force on a segment of string in the interval $[x, x+\epsilon]$ is:

$$F_{\text{vert}} = \underbrace{T(x+\epsilon)\sin[\theta(x+\epsilon,t)] - T(x)\sin[\theta(x,t)]}_{\text{Total Vertical Tension}} + \underbrace{\epsilon\rho(x)Q(x,t)}_{\text{External Force}},$$

where $Q(x,t)$ is an external acceleration (e.g., caused by gravity). Equating the two sides and dividing by ϵ we have:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{T(x+\epsilon)\sin[\theta(x+\epsilon,t)] - T(x)\sin[\theta(x,t)]}{\epsilon} + \rho(x)Q(x,t).$$

Taking the limit as $\epsilon \to 0$ yields the exact force on a piece of the string:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \lim_{\epsilon\to 0}\frac{T(x+\epsilon)\sin[\theta(x+\epsilon,t)] - T(x)\sin[\theta(x,t)]}{\epsilon} + \rho(x)Q(x,t).$$

The Newton quotient can be written as:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left\{T(x,t)\sin[\theta(x,t)]\right\} + \rho(x)Q(x,t). \tag{4.1}$$

Momentarily consider $u(x,t)$ as just a function of x. Then at x, the slope of the tangent line is precisely equal to $\tan[\theta(x)]$. That is:

$$\frac{\partial u}{\partial x} = \frac{\sin(\theta)}{\cos(\theta)}.$$

For θ *small*, we may assume that:

$$\cos(\theta) \approx 1$$

and therefore:

$$\frac{\partial u}{\partial x} \approx \sin[\theta(x,t)].$$

Substituting this (approximation) into Eq. (4.1) yields:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left\{T(x,t)\frac{\partial u}{\partial x}\right\} + \rho(x)Q(x,t). \tag{4.2}$$

Assume $T(x,t) = T_0$ and $\rho(x) = \rho_0$ are constant across the entire string and over time, then we have:

$$\rho_0\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \rho_0 Q(x,t).$$

Dividing through by density yields:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

where $Q(x,t)$ is extrinsic vertical acceleration. If we ignore this by (e.g.) assuming our string is in a vacuum, we have the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2}.$$

Assuming SI units, we can investigate the units of T_0/ρ_0:

$$\frac{kg\, m/s^2}{kg/m} = \frac{m^2}{s^2},$$

or velocity squared. Thus let:

$$\frac{T_0}{\rho_0} = c^2. \tag{4.3}$$

We obtain the usual wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

2. The Acoustic & Electro-Magnetic Waves

2.1. Acoustic Waves.

Remark 4.3. Sound is caused by the coherent motion of molecules in matter. We derive a the acoustic wave equation, which describes the propagation of sound through matter.

Remark 4.4. Logan [Log14] has a derivation of the acoustic wave equation that accomplishes the job. A better derivation is by Feynman [FLS11], but (frankly) the equation numbering is a bit dubious for understanding. However, this may have been intentional to encourage close reading by the students. The following derivation is an adapted form of Feynman's derivation.

Derivation 4.5. Consider a medium (e.g. gas or liquid) in a pipe of constant cross sectional area and suppose a sound (which we leave intentionally vague) is passing through the medium. We investigate the displacement of the medium from the perspective of Newton's Second Law of motion. The situation is illustrated in Fig. 4.2.

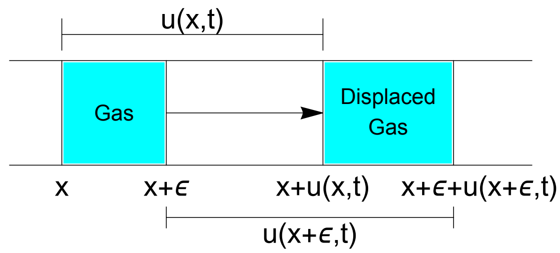

FIGURE 4.2. The propagation of sound through a medium (gas or liquid) causes the displacement of the medium itself.

Density and Pressure. Prior to the arrival of the sound, the medium is in a state of equilibrium with equilibrium pressure P_0 and density ρ_0. If we assume that pressure and density are related by a function f so that $P_0 = f(\rho_0)$ then for a very small change in density (which will be caused by the sound) we have the first order Taylor approximation:

$$P_0 + P_\epsilon = f(\rho_0) + f'(\rho_0)\rho_\epsilon.$$

Here P_ϵ is a tiny change in pressure and ρ_ϵ is a tiny change in density. We use this notation because we are reserving Δ for the Laplacian. But since we assumed $P_0 = f(\rho_0)$, we can write:

$$P_\epsilon = f'(\rho_0)\rho_\epsilon. \tag{4.4}$$

Following [FLS11], let $\kappa = f'(\rho_0)$ be the constant of proportionality.

Displacement of Atoms. Referring to Fig. 4.2, consider a small slice of the medium between x and $x + \epsilon$. It will be displaced by the sound by an amount given by $u(x, t)$. The atoms of the medium at position x are shifted to $x + u(x, t)$ and the atoms of the medium at position $x + \epsilon$ are shifted to the position $x + \epsilon + u(x + \epsilon, t)$.

Density Change. Before the sound, the total mass (per constant area of the pipe cross section) of the medium (gas) between x and $x + \epsilon$ can be computed as:

$$m = \epsilon\rho_0. \tag{4.5}$$

After the sound, the volume may have changed, but the mass remains constant. Assuming a new density ρ (as a result of a sound disturbance) we have:

$$m = \rho\left\{[x + \epsilon + u(x + \epsilon, t)] - [x + u(x, t)]\right\}.$$

Simplifying we have:

$$m = \rho\left\{\epsilon + [u(x + \epsilon, t) - u(x, t)]\right\}.$$

By a Newton quotient argument, for small ϵ, we know:

$$\epsilon\frac{\partial u}{\partial x} \approx u(x + \epsilon, t) - u(x, t).$$

Therefore we have:

$$m = \rho\left(\epsilon + \epsilon\frac{\partial u}{\partial x}\right). \tag{4.6}$$

Setting Eq. (4.5) equal to Eq. (4.6) we have:

$$\epsilon\rho_0 = \epsilon\rho + \rho\epsilon\frac{\partial u}{\partial x}. \tag{4.7}$$

Recall that $\rho = \rho_0 + \rho_\epsilon$ (here ρ_ϵ is the small change in density caused by the sound). We can divide Eq. (4.7) by ϵ and substitute in this expression for ρ to obtain:

$$\rho_0 = \rho_0 + \rho_\epsilon + (\rho_0 + \rho_\epsilon)\frac{\partial u}{\partial x}.$$

If we assume $\rho_\epsilon\partial_x u \approx 0$, then we have an expression for the change in density:

$$\rho_\epsilon = -\rho_0\frac{\partial u}{\partial x}. \tag{4.8}$$

Apply Newton's 2nd Law of Motion. The acceleration of the thin slice of the the medium (gas) is precisely:

$$a = \frac{\partial^2 u}{\partial t^2}.$$

By Newton's second law, the total pressure (force per unit area) on this thin slide of medium is then:

$$\rho_0\epsilon\frac{\partial^2 u}{\partial t^2}, \tag{4.9}$$

where the mass (per unit area) comes from Eq. (4.5). Assuming the sound (and hence flux) is moving left-to-right, the net pressure (force per unit area) is then:

$$P(x, t) - P(x + \epsilon, t) = [P_0 + P_\epsilon(x, t)] - [P_0 + P_\epsilon(x + \epsilon, t)] = \\ P_\epsilon(x, t) - P_\epsilon(x + \epsilon, t) = -\frac{\partial P_\epsilon}{\partial x}\epsilon, \tag{4.10}$$

where P_ϵ is our change in pressure and we assumed P_0 was an equilibrium pressure. Equating Eq. (4.9) and Eq. (4.10), which are both the force (per area) we have:

$$\rho_0 \epsilon \frac{\partial^2 u}{\partial t^2} = -\frac{\partial P_\epsilon}{\partial x}\epsilon.$$

Dividing by ϵ we have:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\frac{\partial P_\epsilon}{\partial x}. \tag{4.11}$$

Putting it Together. From Eq. (4.4) we know $P_\epsilon = \kappa\rho_\epsilon$. From Eq. (4.8) we know $\rho_\epsilon = -\rho_0 \partial_x u$. Substituting those into Eq. (4.11) we have:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x}\left(-\kappa\rho_0 \frac{\partial u}{\partial x}\right).$$

Dividing by ρ_0 and simplifying yields:

$$\frac{\partial^2 u}{\partial t^2} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

This is a variant of the acoustic wave equation, describing the change in displacement of molecules as a result of sound.

Remark 4.6. It is worth noting the acoustic wave equation is normally written in terms of pressure rather than displacement. In that case, just u is pressure and sound is realized as a *pressure wave*, rather than a displacement wave. However, Feynman's derivation is very beautiful and is worth sharing.

2.2. Electromagnetic Waves.

Remark 4.7. Waves occur frequently in nature. We show two other physical derivations of the wave equation to illustrate both its versatility and the process of extracting PDE's from physical phenomena. We start with a lemma.

Lemma 4.8. *Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then:*

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f} \tag{4.12}$$

$\square$

Derivation 4.9 (Electro-Magnetic Wave Equation)**.** Recall Maxwell's equations with no source charge:

$$\begin{cases} \operatorname{div}(\mathbf{E}) = 0 \\ \operatorname{div}(\mathbf{B}) = 0 \\ \operatorname{curl}(\mathbf{E}) = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \operatorname{curl}(\mathbf{B}) = \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In the natural units, we can set $\mu_0\epsilon_0 = 1$, but we leave them for now. The constant μ_0 is the permeability of the vacuum while the constant ϵ_0 is the permittivity of the vacuum. Applying Lemma 4.8 to the third equation yields:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

Maxwell's first laws yield: $\operatorname{div}(\mathbf{E}) = 0$ which is the same as $\nabla \cdot \mathbf{E} = 0$. So our expression becomes:

$$-\nabla^2 \mathbf{E} = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right).$$

We can cancel negatives and interchange the order of differentiation to obtain:

$$\nabla^2 \mathbf{E} = \frac{\partial}{\partial t}\left(\nabla \times \mathbf{B}\right).$$

Now use Maxwell's fourth law to see:

$$\nabla^2 \mathbf{E} = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

We conclude that:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0 \mu_0} \nabla^2 \mathbf{E}. \tag{4.13}$$

This is a wave equation applied to a vector field (rather than a scalar field), making it slightly more complicated than the ordinary wave equation of a scalar field. A similar equation can be found for the magnetic field. We know the propagation speed is:

$$c^2 = \frac{1}{\epsilon_0 \mu_0}.$$

Thus we can compute the speed of electromagnetic wave propagation (speed of light) in a vacuum in SI units as:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx 299,792,458\, m/s.$$

3. Separation of Variables Solution

Proposition 4.10 (String with Fixed Ends)**.** *Consider the one-dimensional wave equation with Dirichlet boundary conditions:*

$$\begin{aligned}
&u_{tt} = c^2 u_{xx}\\
&u(0,t) = u(L,t) = 0\\
&u(x,0) = f(x)\\
&u_t(x,0) = g(x).
\end{aligned}$$

Then the solution to this PDE is given by:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right), \tag{4.14}$$

where a_n is given by Lemma 3.20 as:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

and b_n is given by:

$$b_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

□

EXERCISE 34. Read Derivation 4.12 then prove Proposition 4.10.

Example 4.11. Consider the wave equation with Dirichlet boundary conditions on the interval $[0, \pi]$ and let:

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \leq \pi. \end{cases}$$

Suppose $g(x) = 0$; i.e., if we are modeling the string, it is initially at rest. Then: $b_n = 0$ and we must only compute a_n.

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx =$$
$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\,dx = \begin{cases} \frac{4}{n^2\pi}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we can write:

$$u(x,t) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{(2n-1)^2\pi}\cos\left[(2n-1)ct\right]\sin\left[(2n-1)x\right]$$

Time snapshots of the resulting solution with $c = 1$ are shown in Fig. 4.3.

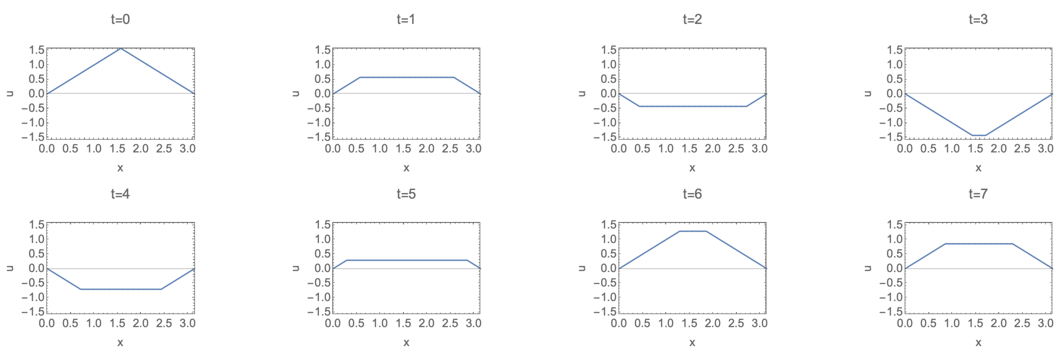

FIGURE 4.3. A solution to the wave equation is illustrated assuming $c = 1$. This models a string plucked into a triangle and then released.

Derivation 4.12. Consider the wave equation on the interval $[0, L]$ with Neumann boundary conditions. This derivation is a bit harder than the case with Dirichlet boundary conditions, which is left as an exercise. The wave equation is second order in t as well as x, so our initial conditions must specify both $u(x, 0)$ and $u_t(x, 0)$. Thus our fully specified problem is:

$$u_{tt} = c^2 u_{xx}$$
$$u_x(0,t) = u_x(L,t) = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x).$$

We approach this with separation of variables again. Let:

$$u(x,t) = v(t)w(x).$$

Then the boundary conditions become:

$$\begin{aligned}
w'(0) &= 0 \\
w'(L) &= 0.
\end{aligned}$$

Differentiating (as before) we have:

$$v''(t)w(x) = c^2 v(t)w''(x) \implies \frac{1}{c^2}\frac{v''(t)}{v(t)} = \frac{w''(x)}{w(x)}.$$

As before the left-hand-side is only a function of t while the right-hand-side is only a function of x. This yields the equations:

$$\begin{aligned}
\frac{1}{c^2}\frac{v''}{v} &= -\lambda \\
\frac{w''}{w} &= -\lambda,
\end{aligned}$$

where λ is a constant and the sign is chosen to be consistent with Lemmas 2.15, 2.18 and 2.21. The second equation provides the boundary value problem:

$$\begin{aligned}
w'' + \lambda w &= 0 \\
w'(0) &= 0 \\
w'(L) &= 0.
\end{aligned}$$

We know from Lemma 2.15 this has solutions:

$$w_n(x) = a_n \cos(n\pi x/L) \tag{4.15}$$

with eigenvalues $\lambda = (n\pi/L)^2$ or $w_0(x) = a_0$ for $\lambda = 0$.

Case I: ($\lambda > 0$) Consider the case when $\lambda > 0$. We now turn our attention to the problem for $v(t)$, which on simplification becomes:

$$v'' + \left(\frac{c^2 n^2 \pi^2}{L^2}\right) v = 0.$$

If we assume $c > 0$, then from the proof of Lemma 2.15 (or Lemma 2.18) we know the solutions have form:

$$v_n(t) = A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right). \tag{4.16}$$

Case II: ($\lambda = 0$) On the other hand, when $\lambda = 0$, the problem for $v(t)$ is:

$$v'' = 0$$

From the proof Lemma 2.15, we know the solutions have form:

$$v_0(t) = C_0 + C_1 t. \tag{4.17}$$

Combining Solutions: Combining Eq. (4.15) with Eq. (4.16) we have:

$$u_n(x,t) = v_n(t)w_n(t) = a_n \cos\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

When $\lambda = 0$, we combine Eq. (4.17) with the solution $w_0(x) = a_0$ to obtain:

$$u_0(x,t) = a_0 + c_1 t.$$

Summing up all solutions we see:

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty} a_n \cos\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right). \tag{4.18}$$

We need only find the values of a_0, c_1, a_n and b_n. Evaluating Eq. (4.18) at $t = 0$ we have:

$$u(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

because $\cos(0) = 1$ and $\sin(0) = 0$. Therefore, we must find a Fourier cosine series expansion of $f(x)$ on the interval $[0, L]$. The coefficients a_0 and a_n are given in Lemma 3.25.

Turning now to the second initial condition, we differentiate $u(x,t)$ term-by-term with respect to t to obtain:

$$u_t(x,t) = b_0 + \sum_{n=1}^{\infty} -a_n \frac{cn\pi}{L}\sin\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + b_n\frac{cn\pi}{L}\cos\left(\frac{cn\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

Now evaluating at $t = 0$:

$$u_t(x,0) = g(x) = b_0 + \sum_{n=1}^{\infty} b_n \frac{cn\pi}{L}\cos\left(\frac{n\pi x}{L}\right).$$

We seek a Fourier cosine series for the function $h(x) = Lg(x)/cn\pi$. Therefore we have:

$$b_n = \frac{2}{L}\int_0^L h(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{L}\int_0^L \frac{L}{cn\pi} g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

Thus we conclude:

$$b_n = \frac{2}{cn\pi}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

The coefficient b_0 is also given in Lemma 3.25 and is:

$$b_0 = \frac{1}{L}\int_0^L g(x)\,dx.$$

Remark 4.13. Solutions to the wave equation with Neumann boundary conditions have the interesting property that they can become unbounded. This is a result of the $b_0 t$ term that appears in the solution. This is clearly a non-physical solution.

Exercise 35. Find a non-trivial example of a wave equation with homogeneous Neumann boundary conditions that does not become unbounded. Hint: The answer $g(x) = 0$ is not non-trivial.]

4. Solutions to the Wave Equation Conserve Energy

Remark 4.14. In this section and the next we are going to analyze the one-dimensional wave equation on the real line. That is, we are going to momentarily ignore boundary conditions. For simplicity, we assume that $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are zero outside a region around 0. That is, there is some place (perhaps far away) from the origin where the string has zero initial velocity and is not displaced and continues that way out to infinity. This will ensure the integrals we take below are finite. Moreover, this also ensures that $u_t(x,t)$ goes to zero as $x \to \pm\infty$. This will be important in a moment.

Remark 4.15. The goal of this short section is to show that solutions to the wave equation exhibit conservation of energy with an appropriate (and somewhat circular) definition of potential energy. This derivation is taken from [**Str08**].

Definition 4.16 (Kinetic Energy)**.** Recall from physics, if an object of mass m is moving with speed v, then its kinetic energy is $\frac{1}{2}mv^2$. For the moving string, we have:

$$E_k = \int_{-\infty}^{\infty} \rho_0 \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$$

Derivation 4.17. Consider the time derivative of the kinetic energy:

$$\frac{dE_k}{dt} = \frac{d}{dt}\left[\rho_0 \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx\right] = \rho_0 \int u_t \cdot u_{tt}\, dx,$$

where the equality on the right-hand-side is achieved by passing the derivative through the integral and applying the chain rule. Use the wave equation to compute:

$$\frac{dE_k}{dt} = \rho_0 \int_{-\infty}^{\infty} c^2 u_t u_{xx}\, dx.$$

For simplicity, we use the derivation of the wave equation from the string. From Eq. (4.3), we can write:

$$\frac{dE_k}{dt} = T_0 \int_{-\infty}^{\infty} u_t u_{xx}\, dx.$$

Now apply integration by parts. Let $dv = u_{xx}\, dx$ and $w = u_t$. Then $v = u_x$ and $dw = w_x\, dx = u_{tx}\, dx$. We conclude:

$$\frac{dE_k}{dt} = T_0 u_t u_x\big|_{-\infty}^{\infty} - T_0 \int_{-\infty}^{\infty} u_x u_{tx}\, dx.$$

By our assumption on $g(x)$, we know that:

$$T_0 u_t u_x\big|_{-\infty}^{\infty} = 0.$$

We also note that:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx\right] = u_x u_{tx}.$$

So we can write:

$$\frac{dE_k}{dt} = -\frac{d}{dt}\left[T_0 \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx\right]$$

Define potential energy as:

$$U = T_0 \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Then the total system energy is:

$$E = E_k + U = \frac{1}{2}\int_{-\infty}^{\infty} \rho_0 \left(\frac{\partial u}{\partial t}\right)^2 + T_0 \left(\frac{\partial u}{\partial x}\right)^2 \, dx, \tag{4.19}$$

and we have proved that:

$$\frac{dE}{dt} = \frac{dE_k}{dt} + \frac{dU}{dt} = -\frac{dU}{dt} + \frac{dU}{dt} = 0.$$

This is a conservation of energy argument and illustrates that E is a conserved quantity in this system.

Remark 4.18. The physics of the potential energy term are a little clearer if you realize that $\frac{1}{2}u_x^2$ is approximately the displacement of the string as a result of being pulled out of its equilibrium position. Thus we are really computing force (tension) times distance.

Remark 4.19. The fact that energy is conserved should not be a surprise (if we put our physicists hats on). There is no friction in this system and we are ignoring any external forces (like gravity). Therefore, energy cannot dissipate out as heat and therefore it must be conserved.

5. Deriving D'Almbert's Solution

Proposition 4.20. *Consider the wave equation on $\mathbb{R}$ with no boundary conditions and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. That is:*

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x). \end{aligned}$$

Then:

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\, dz \tag{4.20}$$

solves this PDE.

Remark 4.21. Eq. (4.20) is called D'Almbert's Formula and it follows from Proposition 1.52, but we will proof it from scratch.

Proof. We will change variables. Define:

$$\xi = x - ct \tag{4.21}$$

$$\eta = x + ct. \tag{4.22}$$

This change of variables is invertible since:

$$x = \frac{1}{2}(\eta + \xi) \tag{4.23}$$

$$t = \frac{1}{2c}(\eta - \xi). \tag{4.24}$$

Define:

$$v(x - ct, x + ct) = v(\xi, \eta) = u\left(\frac{\eta + \xi}{2}, \frac{\eta - \xi}{2c}\right) = u(x,t).$$

Using $u(x,t) = v(\xi,\eta)$, we differentiate:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial v}{\partial \xi} + c\frac{\partial v}{\partial \eta} = c\left(\frac{\partial v}{\partial \eta} - \frac{\partial v}{\partial \xi}\right).$$

By the same argument:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}.$$

We could apply the chain rule again to find the second derivatives, but we can now use an operator trick. Note when it comes to ∂_t applied to v, we see that:

$$\frac{\partial}{\partial t} \equiv c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right).$$

Likewise:

$$\frac{\partial}{\partial x} \equiv \left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right).$$

Then:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)\left[c\left(\frac{\partial v}{\partial \eta} - \frac{\partial v}{\partial \xi}\right)\right].$$

Expand the right-hand-side just like a binomial in algebra to see:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial \eta^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \xi^2}\right).$$

By the same logic:

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right)\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) = \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}.$$

Recall (see Example 1.66) that the wave equation is defined in terms of the D'Almbertian operator:

$$\Box u = 0, \quad \Box = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}.$$

Then:

$$\Box u = 0 \iff c^2\left(\frac{\partial^2 v}{\partial \eta^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \xi^2}\right) - c^2\left(\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right) = 0$$

This implies:

$$-4c^2\frac{\partial^2 v}{\partial \xi \partial \eta} = 0,$$

or more simply:

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0.$$

Let $w = \partial_\eta v$. Then we have:

$$\frac{\partial w}{\partial \xi} = 0.$$

This implies that w is only an arbitrary function of η (i.e., has no dependence on ξ). That is $w = p(\eta)$. But then we have the equation:

$$w = \frac{\partial v}{\partial \eta} = p(\eta).$$

Then integrating we have:

$$v = \int p(\eta)\, d\eta = Q(\xi) + P(\eta),$$

where the *constant of integration* is $Q(\xi)$, which is a function of ξ (i.e., is not dependent on η). The function P is just the formal anti-derivative of $p(\eta)$. Thus we conclude:

$$u(x,t) = Q(x - ct) + P(x + ct). \tag{4.25}$$

This is just Eq. (1.27) with different function names.
Initial Conditions: Applying the initial conditions at $t = 0$, we see that:

$$Q(x) + P(x) = f(x) \tag{4.26}$$

$$-cQ'(x) + cP'(x) = g(x). \tag{4.27}$$

Differentiate the first equation and multiply by c to obtain:

$$cQ'(x) + cP'(x) = cf'(x) \tag{4.28}$$

$$-cQ'(x) + cP'(x) = g(x). \tag{4.29}$$

Adding Eqs. (4.28) and (4.29):

$$2cP'(x) = cf'(x) + g(x)$$

We can integrate to see:

$$P(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\, dz.$$

But then since $Q(x) + P(x) = f(x)$, we condlude:

$$Q(x) = f(x) - P(X) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\, dz.$$

Recall the solution is given by Eq. (4.25). Therefore, we can compute:

$$u(x,t) = Q(x - ct) + P(x + ct) =$$
$$\left(\frac{1}{2}f(x - ct) - \frac{1}{2c}\int_0^{x-ct} g(z)\, dz.\right) + \frac{1}{2}f(x + ct) + \frac{1}{2c}\int_0^{x+ct} g(z)\, dz.$$

Combining terms we have:

$$u(x,t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\, dz.$$

because we recall from calculus that:

$$-\frac{1}{2c}\int_0^{x-ct} g(z)\, dz = \frac{1}{2c}\int_{x-ct}^0 g(z)\, dz$$

This completes the proof. □

EXERCISE 36. Compute the solution to the wave equation on $\mathbb{R}$ assuming an unknown velocity of c and initial conditions:

$$u(x,0) = \begin{cases} x+\pi & \text{if } -\pi \leq x < 0 \\ \pi - x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases} \qquad u_t(x,0) = \begin{cases} -1 & \text{if } -\pi \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

EXERCISE 37. Assume $c = 1$ and replace the the initial condition in the previous problem with $u_t(x,0) = 0$; i.e., assume no initial vertical velocity in a plucked string. Compute the total energy in the string assuming $\rho_0 = T_0 = 1$. [Hint: Since energy is conserved, it suffices to compute the derivatives in Eq. (4.19) at $t = 0$. Take the integral to find the value.]

CHAPTER 5

The Laplace Equation

1. Electrostatic Origin of the Laplace Equation

Remark 5.1. Recall from Remark 1.47 that one interpretation of the Laplace equation is that solutions represent the stationary (long-run) behavior of a heated object (e.g., a disk) that has a specific heat profile on the boundary (Dirichlet boundary condition) or heat exchange profile (Neumann boundary condition).

Remark 5.2. Another classic place the Laplace equation emerges is in electrostatics. We provide a quick derivation.

Derivation 5.3. Let u be a scalar electric potential; i.e., a scalar function that returns the amount of work required to move a charge from a fixed reference point to a point (x, y, z). By definition we have:

$$\vec{\mathbf{E}} = -\operatorname{grad} u = -\nabla u.$$

From the differential form of Gauss' law (in CGS), we have:

$$\operatorname{div} \mathbf{E} = \nabla \cdot \mathbf{E} = 4\pi\rho,$$

where ρ is the charge density in free space *inside* a region of interest. Combining these equations together yields:

$$-\operatorname{div} \operatorname{grad} u = -\nabla \cdot \nabla u = 4\pi\rho.$$

Now assume the charge density is zero inside a region of interest (but may be non-zero on the boundary of the region). Then we obtain:

$$\nabla^2 u = 0,$$

which is the Laplace equation. Solutions of the Laplace equation will describe the electrostatic potential within a region of interest (like a sphere) assuming information about the potential function on the boundary and assuming there are no internal charges within the region of interest.

2. The Laplace Equation on a Rectangle

Remark 5.4. In this section, we will derive a solution to the Laplace equation on a rectangle assuming Dirichlet boundary conditions on three of four sides have a Dirichlet boundary condition and the fourth side has a functional boundary condition. In this way, this side will play the role of an initial condition though there is no temporal variable.

Remark 5.5. This does not seem to be very general. However, we will show at the conclusion of the derivation how to generalize this to an arbitrary problem.

Derivation 5.6. Consider a rectangular sheet of (infinitely thin) metal located between $0 \leq x \leq L$ and $0 \leq y \leq H$. We have the following PDE:

$$\begin{aligned}
&u_{xx} + u_{yy} = 0\\
&u(0, y) = f(y)\\
&u(L, y) = 0\\
&u(x, 0) = 0\\
&u(x, H) = 0
\end{aligned}$$

This is illustrated in Fig. 5.1. We approach this with separation of variables again. Let:

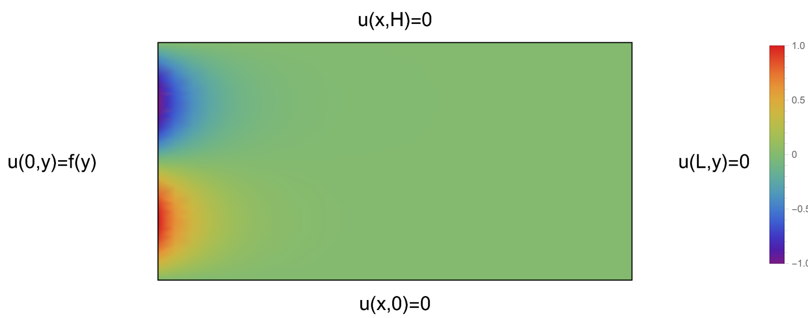

FIGURE 5.1. The problem setup for the Laplace equation on a rectangle is illustrated.

$$u(x, y) = v(x)w(y).$$

Then the boundary conditions become:

$$\begin{aligned}
v(L) &= 0\\
w(0) &= 0\\
w(H) &= 0.
\end{aligned}$$

Differentiating we have (as before):

$$v''(x)w(y) + v(x)w''(y) = 0 \iff -\frac{v''}{v} = \frac{w''}{w}.$$

Again the fact that the left-hand-side is only a function of x while the right-hand-side is only a function y yields the two equations:

$$\begin{aligned}
-\frac{v''}{v} &= -\lambda\\
\frac{w''}{w} &= -\lambda,
\end{aligned}$$

where λ is a constant and the sign is chosen to be consistent with Lemmas 2.15, 2.18 and 2.21. Considering the boundary value problem:

$$\begin{aligned}
&w'' + \lambda w = 0\\
&w(0) = 0\\
&w(H) = 0.
\end{aligned}$$

We know from Lemma 2.15 this has solutions $w_n(y) = b_n \sin(n\pi y/H)$ with eigenvalues $\lambda = (n\pi/H)^2$. Given this information, the second boundary value problem is:

$$v'' - \left(\frac{n\pi}{H}\right)^2 v = 0$$

From Exercise 18 we know the general solution is:

$$v(x) = A_1 \cosh\left[\left(\frac{n\pi}{H}\right)(x-L)\right] + A_2 \sinh\left[\left(\frac{n\pi}{H}\right)(x-L)\right].$$

Evaluating $v(L)$ and noting that $\sinh(0) = 0$ and $\cosh(0) = 1$ we see $A_1 = 0$ (to cancel out the hyperbolic cosine term) leaving a solution:

$$v(x) = A_2 \sinh\left[\left(\frac{n\pi}{H}\right)(x-L)\right].$$

This seems a little arbitrary, but we note that $\cosh(x)$ is positive everywhere. Meaning there is no way to choose A_1 so that $A_1 \cosh(0) = 0$. We illustrate this in Fig. 5.2. Now combining

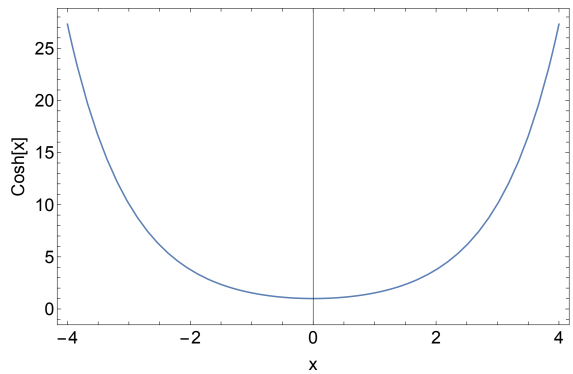

FIGURE 5.2. The hyperbolic cosine function is an everywhere positive function with minimum at 0.

these solutions together we see the solution to the Laplace equation (ignoring one of the boundary conditions) is:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left[\left(\frac{n\pi}{H}\right)(x-L)\right] \sin\left(\frac{n\pi}{H}y\right). \tag{5.1}$$

We now use the boundary condition $u(0,y) = f(y)$ to determine the unknown coefficients a_n.

Evaluating $u(0,y)$ we have:

$$u(0,y) = f(y) = \sum_{n=1}^{\infty} a_n \sinh\left[-\left(\frac{n\pi L}{H}\right)\right] \sin\left(\frac{n\pi}{H}y\right).$$

To clear up the equation, let:

$$b_n = a_n \sinh\left[-\left(\frac{n\pi L}{H}\right)\right].$$

Then we have:

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{H}y\right).$$

This is just a Fourier sine series and we know from Lemma 3.20 we have:

$$b_n = \frac{2}{H}\int_0^H f(y)\sin\left(\frac{n\pi}{H}y\right)\,dy.$$

From this we deduce:

$$a_n = \frac{1}{\sinh\left[-\left(\frac{n\pi L}{H}\right)\right]}\frac{2}{H}\int_0^H f(y)\sin\left(\frac{n\pi}{H}y\right)\,dy.$$

Remark 5.7. Derivation 5.6 seems very limited. It would be more useful to have a solution for the boundary value problem:

$$\begin{aligned} &u_{xx} + u_{yy} = 0\\ &u(0,y) = f_l(y)\\ &u(L,y) = f_r(y)\\ &u(x,0) = g_b(x)\\ &u(x,H) = g_t(x) \end{aligned}$$

Fortunately, we can generalize the result in Proposition 5.8.

Proposition 5.8. *Suppose that $u_l(x,y)$, $u_r(x,y)$, $u_b(x,y)$ and $u_t(x,y)$ solve the problems:*

$$\begin{cases} \Delta u_l = 0\\ u_l(0,y) = f_l(y)\\ u_r(L,y) = u_b(x,0) = u_t(x,H) = 0 \end{cases} \qquad \begin{cases} \Delta u_r = 0\\ u_r(L,y) = f_r(y)\\ u_l(0,y) = u_b(x,0) = u_t(x,H) = 0 \end{cases}$$

$$\begin{cases} \Delta u_b = 0\\ u_b(x,0) = g_b(x)\\ u_t(x,H) = u_l(0,y) = u_r(L,y) = 0 \end{cases} \qquad \begin{cases} \Delta u_t = 0\\ u_t(x,H) = g_t(x)\\ u_b(x,0) = u_l(0,y) = u_r(L,y) = 0. \end{cases}$$

Then:

$$u(x,y) = u_l(x,y) + u_r(x,y) + u_b(x,y) + u_t(x,y)$$

solves the general Laplace equation on the rectangle:

$$\begin{aligned} &\Delta u = 0\\ &u(0,y) = f_l(y)\\ &u(L,y) = f_r(y)\\ &u(x,0) = g_b(x)\\ &u(x,H) = g_t(x). \end{aligned}$$

Proof. Linearity implies that:

$$\Delta u = \Delta(u_l + u_r + u_b + u_t) = 0.$$

We consider one boundary:

$$u(0,y) = u_l(0,y) + u_r(0,y) + u_b(0,y) + u_r(0,y) = f_l(y) + 0 = f_l(y).$$

The other boundary conditions follow in the same way. $\square$

Example 5.9. Consider the Laplace equation on $[0,1]\times[0,1]$ with the boundary conditions:

$$\begin{aligned}
f_b(x,0) &= 3+\sin(\pi x)\\
f_t(x,1) &= 2-\sin(2\pi x)\\
f_l(0,y) &= 0\\
f_r(1,y) &= 0
\end{aligned}$$

Using the result from Eq. (5.1), we can write the form of $u_b(x,y)$ and $u_t(x,y)$. Remember we will have to swap x and y however, because in our derivation we have a boundary condition in terms of y and not x. Likewise we will have $L=0$ and $L=1$.

$$\begin{aligned}
u_b(x,y) &= \sum_{n=1}^{\infty} a_n \sinh\left[n\pi(y-1)\right]\sin\left(n\pi x\right)\\
u_t(x,y) &= \sum_{n=1}^{\infty} a_n \sinh\left(n\pi y\right)\sin\left(n\pi x\right).
\end{aligned}$$

In the first case, we have:

$$a_n = \frac{2}{\sinh\left(-n\pi\right)}\int_0^1 \left(3+\sin(\pi x)\right)\sin\left(n\pi x\right)\,dx = \frac{2}{\sinh\left(-n\pi\right)}\cdot\begin{cases}\frac{1}{2}+\frac{6}{\pi} & \text{if } n=1\\ \frac{6}{n\pi} & \text{if } n>1 \text{ and } n \text{ is odd}\\ 0 & \text{otherwise.}\end{cases} \tag{5.2}$$

In the second case we have:

$$a_n = \frac{2}{\sinh\left(n\pi\right)}\int_0^1 \left(2-\sin(2\pi x)\right)\sin\left(n\pi x\right)\,dx = \frac{2}{\sinh\left(n\pi\right)}\cdot\begin{cases}\frac{4}{n\pi} & \text{if } n \text{ is odd}\\ -\frac{1}{2} & \text{if } n=2\\ 0 & \text{otherwise.}\end{cases}$$

Using 300 Fourier coefficients, we can approximate the solution (reasonably enough). We illustrate our approximation and an exact solution using Mathematica in Fig. 5.3.

3. Review of Polar Coordinates

Remark 5.10. In this section we derive some preliminary results required for an analysis of the Laplace equation on the disk.

Definition 5.11. If (x,y) is a point in $\mathbb{R}^2$ expressed in Cartesian coordinates then in *polar coordinates* it is expressed as the ordered pair (r,θ), where:

$$r = \sqrt{x^2+y^2} \tag{5.3}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{5.4}$$

The inverse relationship is:

$$x = r\cos(\theta) \tag{5.5}$$

$$y = r\sin(\theta) \tag{5.6}$$

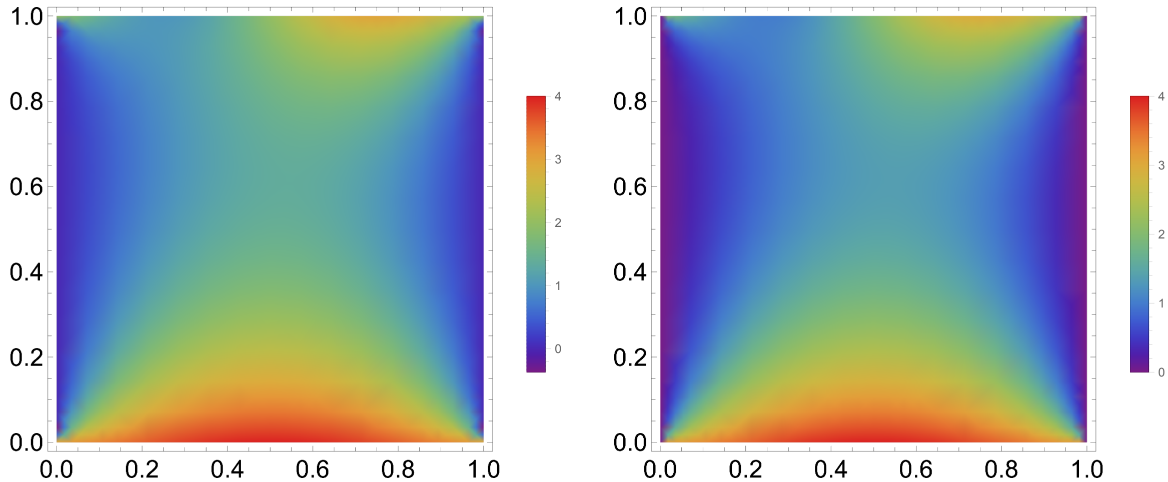

FIGURE 5.3. (Left) An exact solution to the example Laplace equation produced using Mathematica's PDE solver. (Right) An approximation of the solution using 300 Fourier coefficients.

Remark 5.12. The definition for θ in Eq. (5.4) has some difficulties, especially when $x = y = 0$ or even when $x = 0$. These difficulties are usually summarized by using the arctan2 function, which deals with these irregularities very explicitly. We do not have to worry about the irregularities in our derivations. Interested students can investigate the arctan2 function on Wikipedia.

Proposition 5.13. *The Laplacian in polar coordinates is:*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{5.7}$$

PROOF. First note:

$$\frac{dr}{dx} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos(\theta) \quad \text{and} \quad \frac{dr}{dy} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin(\theta).$$

Also,

$$\frac{d\theta}{dx} = -\frac{y}{x^2\left(\frac{y^2}{x^2}+1\right)} = -\frac{y}{r^2} = -\frac{\sin(\theta)}{r} \quad \text{and} \quad \frac{d\theta}{dy} = \frac{1}{x\left(\frac{y^2}{x^2}+1\right)} = \frac{x}{r^2} = \frac{\cos(\theta)}{r}.$$

Let $u(r, \theta)$ be a function. By the chain rule we have:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{dr}{dx} + \frac{\partial u}{\partial \theta}\frac{d\theta}{dx} = u_r\cos(\theta) - \frac{\sin(\theta)}{r}u_\theta. \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{dr}{dy} + \frac{\partial u}{\partial \theta}\frac{d\theta}{dy} = u_r\sin(\theta) + \frac{\cos(\theta)}{r}u_\theta.
\end{aligned}$$

We can write the operators:

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta} \tag{5.8}$$

$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}. \tag{5.9}$$

Applying ∂_x to itself we have:

$$\left(\cos(\theta)\frac{\partial}{\partial r}\right)\left(\cos(\theta)\frac{\partial}{\partial r}\right) = \cos^2(\theta)\frac{\partial^2}{\partial r^2}$$

$$\left(\cos(\theta)\frac{\partial}{\partial r}\right)\left(-\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right) = \frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial}{\partial \theta} - \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta}$$

$$\left(-\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\frac{\partial}{\partial r}\right) = \frac{\sin^2(\theta)}{r}\frac{\partial}{\partial r} - \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta}$$

$$\left(-\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(-\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right) = \frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Adding this together yields:

$$\frac{\partial^2}{\partial x^2} = \cos^2(\theta)\frac{\partial^2}{\partial r^2} + \frac{2\sin(\theta)\cos(\theta)}{r^2}\frac{\partial}{\partial \theta} - \frac{2\sin(\theta)\cos(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\sin^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

We can apply the same process to compute ∂_{yy} to obtain:

$$\frac{\partial^2}{\partial y^2} = \sin^2(\theta)\frac{\partial^2}{\partial r^2} - \frac{2\sin(\theta)\cos(\theta)}{r^2}\frac{\partial}{\partial \theta} + \frac{2\sin(\theta)\cos(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\cos^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Adding and cancelling yields:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

□

Remark 5.14. The next (and last) result presented in this section deals with a special ordindary differential equation that will emerge when we attempt separation of variables on the Laplace equation in polar coordinates.

Derivation 5.15 (Cauchy-Euler Differential Equation)**.** Consider the second order differential equation with non-constant coefficients and unknown function $v(r)$:

$$r^2 v'' + arv' + bv = 0. \tag{5.10}$$

Let $x = \log(r)$. Assume that $v(r) = \phi(x)$ for some unknown function ϕ. Then:

$$\frac{dv}{dr} = \frac{dx}{dr}\frac{d\phi}{dx} = \frac{1}{r}\frac{d\phi}{dx}. \tag{5.11}$$

Differentiating again and using the product rule yields:

$$\frac{d^2v}{dr^2} = -\frac{1}{r^2}\frac{d\phi}{dx} + \frac{1}{r^2}\frac{d^2\phi}{dx^2}, \tag{5.12}$$

where the second term follows by:

$$\frac{1}{r}\left(\frac{d}{dr}\frac{d\phi}{dx}\right) = \frac{1}{r}\left(\frac{d}{dx}\frac{d\phi}{dr}\right) = \frac{1}{r}\left(\frac{d}{dx}\frac{d\Omega}{dr}\right) = \frac{1}{r}\left(\frac{d}{dx}\frac{1}{r}\frac{d\phi}{dx}\right) = \frac{1}{r^2}\frac{d^2\phi}{dx^2}$$

Substitute Eqs. (5.11) and (5.12) in Eq. (5.10) to obtain:

$$\phi'' + (a-1)\phi' + b\phi = 0.$$

This is a second order differential equation with constant coefficients, which has solutions:

$$\phi = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x),$$

where λ_1 and λ_2 are the distinct roots of the characteristic polynomial:

$$s^2 + (a-1)s + b = 0,$$

if such distinct roots exist. If the roots are not distinct, the solution is:

$$\phi = c_1 \exp(\lambda_1 x) + c_2 x \exp(\lambda_1 x).$$

We now substitute $x = \log(r)$ into these solutions. In the case of two distinct solutions to the characteristic polynomial, we have:

$$v = c_1 r^{\lambda_1} + c_2 r^{\lambda_2}.$$

In the case of one solution to the characteristic polynomial we have:

$$v = c_1 r^{\lambda_1} + c_2 \log(r) r^{\lambda_1}.$$

Corollary 5.16. *The solution to the Cauchy-Euler equation:*

$$r^2 v'' + r v' - n^2 v = 0,$$

is:

$$v_n(r) = c_1 r^n + c_2 r^{-n},$$

when $n \neq 0$ and

$$v_0(r) = c_1 + c_2 \log(r),$$

when $n = 0$.

Exercise 38. Prove Corollary 5.16 using Derivation 5.15.

4. Laplace Equation on a Disk

Remark 5.17. We now consider the Laplace Equation on a disk using polar coordinates. Solutions to such an equation are illustrated in Remark 1.47 for a specific boundary condition.

Derivation 5.18. Consider the Laplace equation on the disk of radius R (centered at the origin):

$$\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \\
&u(R,\theta) = f(\theta) \\
&f(r,-\pi) = f(r,\pi)
\end{aligned}$$

We proceed with separation of variables. Let $u(r,\theta) = v(r)w(\theta)$. Then Laplace's equation becomes:

$$v''(r)w(\theta) + \frac{1}{r}v'(r)w(\theta) + \frac{1}{r^2}v(r)w''(\theta) = 0.$$

Separating the variables we have:

$$-\frac{r^2 v''(r) + rv'}{v} = \frac{w''}{w}.$$

As before, we conclude that there is a constant $-\lambda$ so that:

$$\begin{aligned}
&\frac{w''}{w} = -\lambda \\
&\frac{r^2 v''(r) + rv'}{v} = \lambda.
\end{aligned}$$

Analyzing the first equation we have the boundary value problem:

$$\begin{aligned}
&w'' + \lambda w = 0 \\
&w(-\pi) = w(\pi)
\end{aligned}$$

From Lemma 2.21, using $L = \pi$, we know this has solutions when:

$$\lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2.$$

Therefore $w_0(x) = a_0$ and $w_n(x) = a_n \cos(nx) + b_n \sin(nx)$. Substituting this into the equation for $v(r)$ we have the Cauchy-Euler equation:

$$r^2 v'' + rv' - n^2 v,$$

which has solution (from Corollary 5.16)

$$v_n(r) = c_1 r^n + c_2 r^{-n},$$

when $n \neq 0$ and

$$v_0(r) = c_1 + c_2 \log(r),$$

when $n = 0$. Notice that we are free to set c_1 or c_2 and we will have a *big problem* if $c_2 \neq 0$ because r^{-n} grows asymptotically as $r \to 0$. Therefore, if we don't set $c_2 = 0$, we will have a non-physical solution. The same is true when $n = 0$ because $\log(r)$ grows asymptotically (in absolute value) as $r \to 0$.

Combining all the remaining constants we have the solution:

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta). \tag{5.13}$$

We must still determine the coefficients. We know that $u(R, \theta) = f(\theta)$. Thus:

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} R^n a_n \cos(n\theta) + R^n b_n \sin(n\theta).$$

This is just a Fourier series expansion. We have to divide out the constant R^n in our Fourier coefficient formulas on $[-\pi, \pi]$, but doing so we conclude:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \tag{5.14}$$

$$a_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta, \tag{5.15}$$

$$b_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta. \tag{5.16}$$

Example 5.19. Let's find the closed form solution to the Laplace equation first illustrated in Remark 1.47. On the unit disk we have:

$$\begin{aligned}
&\Delta u = 0\\
&u(1,\theta) = \sin(4\theta)\\
&u(r,-\pi) = u(r,\pi).
\end{aligned}$$

Note we have replaced the periodic boundary conditions $u(r, 0) = u(r, 2\pi)$ from Remark 1.47 with slightly different boundary conditions. This will make no difference. Here we have $R = 1$ (which simplifies the math).

We know the general solution is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta).$$

It's a simple matter of computation to see that:

$$a_0 = \int_{-\pi}^{\pi} \sin(4\theta)\,d\theta = 0.$$

Likewise, we know the Fourier series of $\sin(4\theta)$ on the interval $[-\pi, \pi]$ should have no cosine terms and only the term $b_4 \sin(4\theta)$. That is $a_n = 0$ for all n and $b_n = 0$ if $n \neq 4$. Therefore, we compute:

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(4\theta)\,d\theta = 1$$

We conclude that:

$$u(r,\theta) = r^4 \sin(4\theta).$$

A density plot of the solution is shown in Fig. 5.4, which can be compared to Fig. 1.10.

5. Poisson Integral Formula

Remark 5.20. We can transform the unwieldy Fourier solution of the Laplace equation on the disk into a beautiful integral representation. But we need some initial results.

Remark 5.21 (Complex Number)**.** Remember, a complex number is a number of the form $z = x + iy$. Therefore, each complex number can be represented by the Cartesian point (x, y) and thus has a polar form (r, θ) that can be computed from Definition 5.11.

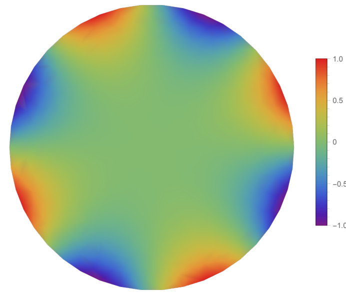

FIGURE 5.4. A density plot on the unit disk of the computed solution to the Laplace equation.

Definition 5.22 (Modulus of a Complex Number)**.** The *modulus* or absolute value of a complex number $z = x + iy$ is:

$$|z| = x^2 + y^2$$

Remark 5.23. Consequently, in polar coordinates, a complex number $x + iy$ has $r = |z| = x^2 + y^2$.

Remark 5.24. The next lemma can be proved[1] using the Taylor series expansions of the functions involed. It's a fun exercise to do at home, if you've never done it or seen it.

Lemma 5.25 (Euler's Theorem)**.** *For real values (r, θ) with $r > 0$ we have:*

$$re^{i\theta} = r\cos(\theta) + ir\sin(\theta).$$

Thus any complex number can be represented in polar form as $re^{i\theta}$. □

Corollary 5.26. *For any θ:*

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

EXERCISE 39. Prove Corollary 5.26

Remark 5.27. The next lemma is a mild generalization of a well known result from calculus.

Lemma 5.28. *If z is a complex number with $|z| < 1$, then:*

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n$$

and the series on the right-hand-side converges. □

Corollary 5.29. *If z is a complex number with $|z| < 1$, then:*

$$\frac{z}{1-z} = z + z^2 + z^3 + \cdots = \sum_{n=1}^{\infty} z^n$$

and the series on the right-hand-side converges. □

Remark 5.30. Note, we have $|z| < 1$ if in its polar representation $z = re^{i\theta}$ we have $r < 1$.

[1] A really formal proof requires convergence results, which we are ignoring.

Derivation 5.31 (Poisson's Integral Formula)**.** To make the symbols easier to follow, replace θ with φ in Eqs. (5.14) to (5.16) and insert those coefficients direction in Eq. (5.13) to obtain:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi + \sum_{n=1}^{\infty}\left(\frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi)\cos(n\varphi)\,d\varphi\right) r^n\cos(n\theta) + \left(\frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi)\sin(n\varphi)\,d\varphi\right) r^n\sin(n\theta). \tag{5.17}$$

Any term with a θ, r, or R is a constant with respect to the integration and so we can move those terms through the integrals to obtain:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi + \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n f(\varphi)\left[\cos(n\varphi)\cos(n\theta)+\sin(n\varphi)\sin(n\theta)\right]\right\} d\varphi. \tag{5.18}$$

Use the identity:

$$\cos(n\theta - n\varphi) = \cos(n\varphi)\cos(n\theta) + \sin(n\varphi)\sin(n\theta) \tag{5.19}$$

to simplify Eq. (5.18) further as:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\left\{\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos[n(\theta-\varphi)]\right\} d\varphi$$

Lastly we can cleverly multiply by 1 to combine the two integrals together[2]:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos[n(\theta-\varphi)]\right\} d\varphi$$

Now apply Corollary 5.26 to obtain:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \frac{e^{in(\theta-\varphi)}+e^{-in(\theta-\varphi)}}{2}\right\} d\varphi \tag{5.20}$$

Focus on the series under the integral and note:

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \frac{e^{in(\theta-\varphi)}+e^{-in(\theta-\varphi)}}{2} = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}e^{i(\theta-\varphi)}\right)^n + \sum_{n=1}^{\infty}\left(\frac{r}{R}e^{-i(\theta-\varphi)}\right)^n. \tag{5.21}$$

Now define the complex numbers

$$z_1 = \frac{r}{R}e^{i(\theta-\varphi)}$$
$$z_2 = \frac{r}{R}e^{-i(\theta-\varphi)}$$

Then:

$$|z_1| = |z_2| = \frac{r}{R} < 1,$$

[2]If you can't see what happened, we needed to have a 2π in the denominator of the fractions multiplying both integrals.

because our disk has radius R (and we know the value of $u(r, \theta)$ on the boundary). We now apply Corollary 5.29 to Eq. (5.21) to see:

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \frac{e^{in(\theta-\varphi)} + e^{-in(\theta-\varphi)}}{2} = 1 + \frac{z_1}{1-z_1} + \frac{z_2}{1-z_2} =$$

$$1 + \frac{\frac{r}{R}e^{i(\theta-\varphi)}}{1-\frac{r}{R}e^{i(\theta-\varphi)}} + \frac{\frac{r}{R}e^{-i(\theta-\varphi)}}{1-\frac{r}{R}e^{-i(\theta-\varphi)}} = 1 + \frac{re^{i(\theta-\varphi)}}{R-re^{i(\theta-\varphi)}} + \frac{re^{-i(\theta-\varphi)}}{R-re^{-i(\theta-\varphi)}}$$

Cross-multiply to obtain:

$$1 + \frac{re^{i(\theta-\varphi)}}{R-re^{i(\theta-\varphi)}} + \frac{re^{-i(\theta-\varphi)}}{R-re^{-i(\theta-\varphi)}} = 1 + \frac{rR\left[e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}\right] + -2r^2}{R^2 - rR\left[e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}\right] + r^2}$$

Adding the terms (by re-writing 1 to have a common denominator) we obtain:

$$1 + \frac{rR\left[e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}\right] + -2r^2}{R^2 - rR\left[e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}\right] + r^2} = \frac{R^2 - r^2}{R^2 + r^2 - rR\left[e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}\right]}.$$

Now apply Corollary 5.26 to obtain:

$$\frac{R^2 - r^2}{R^2 + r^2 - rR\left[e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}\right]} = \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos[(\theta-\varphi)]}.$$

This is called the *Poisson kernel* and we can use it to re-write Eq. (5.20) as:

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos[(\theta-\varphi)]}\,d\varphi,$$

which is called the *Poisson integral formula*. We have proved a theorem.

THEOREM 5.32. *If $f(\theta)$ is a continuous function, then the solution to the Laplace equation with boundary value $f(\theta)$ on the interior of the disk of radius R is given by:*

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos[(\theta-\varphi)]}\,d\varphi.$$

□

Remark 5.33. If you've taken Complex Analysis (Math 421 at PSU), you might recall that in two dimensions, many unusual domains Ω can be mapped conformally to the unit disk. This implies that convenient closed form solutions can be found in two dimension. This is not true in three or more dimensions.

6. Theoretical Results on the Laplace Equation

Remark 5.34. The following theorem, called the mean value theorem, follows immediately from the Fourier series solution or Poisson's integral formula. We will generalize it momentarily.

THEOREM 5.35 (Mean Value Theorem for Laplace Equation). *If $u(r, \theta)$ is a solution to the Laplace equation in a disk with boundary value $f(\theta)$, then:*

$$u(0, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta.$$

That is, the value of u at the center of the disk is equal to the mean value of the value at the boundary. □

EXERCISE 40. Prove Theorem 5.35 both using the Fourier series and the Poisson integral formula. Verify they are the same.

Remark 5.36. The mean value theorem for the Laplace equation actually holds for balls in any dimension, not just $\mathbb{R}^2$. The regular integration is replaced with a (hyper)-surface integral over the ball.We currently only have the machinery to prove it in $\mathbb{R}^2$.

Definition 5.37 (Harmonic Function)**.** If u solves the Laplace equation $\Delta u = 0$, then u is a *harmonic function.*

THEOREM 5.38 (Maximum/Minimum Principle). *Suppose that u is a harmonic function. Then on any domain $\Omega \subseteq \mathbb{R}^2$, u achieves both its maximum value and minimum value on the boundary of Ω.*

Remark 5.39. We will provide a proof sketch for arbitrary dimensions assuming that Theorem 5.35 extends to arbitrary dimensions.

PROOF SKETCH FOR MAXIMUM/MINIMUM PRINCIPLE. Assume u is non-constant on Ω and its boundary. We proceed by contradiction. Assume that there is a point $\mathbf{x}^*$ in the interior of Ω so that $u(\mathbf{x}^*)$ is strictly greater than any other value in Ω (including its boundary). Then choose any ball B with center $\mathbf{x}^*$ contained in Ω. We know that $u(\mathbf{x}^*)$ has value equal to the mean of the boundary value on the ball B. But this cannot be unless every point on the boundary has value $u(\mathbf{x}^*)$, since $\mathbf{x}^*$ is a maximum. By a similar argument, it follows that every point in Ω would have the same value. This is a contradiction and implies that u achieves its maximum on the boundary of Ω. A similar argument proves the minimum principle as well. $\square$

Corollary 5.40. *The solution to the Laplace equation varies continuously with respect to the input data.*

PROOF. Consider two instances of the Laplace equation on a domain Ω:

$$\begin{cases} \Delta u = 0 \\ u(\mathbf{x}) = f(\mathbf{x}) & \mathbf{x} \in \partial\Omega. \end{cases} \qquad \begin{cases} \Delta v = 0 \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \partial\Omega. \end{cases}$$

By linearity $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$ solves the Laplace equation with boundary condition $u(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$. By the maximum/minimum principle we know:

$$\min_{\mathbf{x}\in\partial\Omega} f(\mathbf{x}) - g(\mathbf{x}) \leq w(\mathbf{w}) \leq \max_{\mathbf{x}\in\partial\Omega} f(\mathbf{x}) - g(\mathbf{x}),$$

for all $\mathbf{x} \in \Omega$. Therefore, if:

$$|f(x) - g(x)| < \epsilon,$$

then:

$$|u(\mathbf{x}) - v(\mathbf{x})| < \epsilon.$$

for all $\mathbf{x} \in \Omega$. $\square$

Remark 5.41. The objective of the remainder of this section is to prove that the solution to the Laplace equation is unique (at least in 2 and 3 dimensions). It turns out that this is true in all dimensions. If we do this, then we will have proved that the Laplace equation is well-posed in the sense we described in Chapter 1. To do this, we'll state several preliminary lemmas.

Remark 5.42. The next lemma is just a product rule in vector calculus. (Remember dot products commute.)

Lemma 5.43. *Let $\Omega \subseteq \mathbb{R}^3$ and suppose $u : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function and suppose $\boldsymbol{\varphi} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field (vector valued function). Then:*

$$\operatorname{div}(u\boldsymbol{\varphi}) = u \operatorname{div}(\boldsymbol{\varphi}) + \boldsymbol{\varphi} \cdot \operatorname{grad}(u). \tag{5.22}$$

□

Theorem 5.44 (Green's First Identity). *Let $\Omega \subseteq \mathbb{R}^3$ and let $S = \partial\Omega$ and suppose $u : \mathbb{R}^3 \to \mathbb{R}$ is a scalar C^2 function. Then:*

$$\int_S [u \operatorname{grad}(u)] \cdot \mathbf{n}\, dS = \int_\Omega u\, \Delta u\, d\Omega + \int_\Omega \|\operatorname{grad}(u)\|^2\, d\Omega. \tag{5.23}$$

Here (as we expect) Δu is the Laplacian of u.

Proof. Let $\boldsymbol{\varphi} = \operatorname{div}(u)$. Then from Eq. (5.22), we have:

$$\operatorname{div}(u\boldsymbol{\varphi}) = u \operatorname{div}(\boldsymbol{\varphi}) + \boldsymbol{\varphi} \cdot \operatorname{grad}(u).$$

Apply the divergence theorem (Theorem 2.5) to see:

$$\int_\Omega \operatorname{div}(u\,\boldsymbol{\varphi})\, d\Omega = \int_{\partial\Omega} (u\,\boldsymbol{\varphi}) \cdot \mathbf{n}\, dS.$$

Apply Lemma 5.43 to see the left-hand-side is:

$$\int_\Omega u \operatorname{div}(\boldsymbol{\varphi})\, d\Omega + \int_\Omega \boldsymbol{\varphi} \cdot \operatorname{grad}(u)\, d\Omega = \int_{\partial\Omega} (u\,\boldsymbol{\varphi}) \cdot \mathbf{n}\, dS$$

Now substitute $\boldsymbol{\varphi} = \operatorname{grad}(u)$ to see:

$$\int_\Omega u \operatorname{div}[\operatorname{grad}(u)]\, d\Omega + \int_\Omega \operatorname{grad}(u) \cdot \operatorname{grad}(u)\, d\Omega = \int_{\partial\Omega} [u \operatorname{grad}(u)] \cdot \mathbf{n}\, dS,$$

which can be written

$$\int_\Omega u\, \Delta u\, d\Omega + \int_\Omega \|\operatorname{grad}(u)\|^2\, d\Omega = \int_{\partial\Omega} [u \operatorname{grad}(u)] \cdot \mathbf{n}\, dS,$$

using the fact that the dot product of a vector (field) with itself is its norm squared and the definition of the Laplace operator (see Definition 1.19). □

Remark 5.45. Green's first identity is actually a higher dimensional analog of integration by parts. To see this, consider the one dimensional analog of the term with the Laplacian:

$$\int_a^b u(x)\, u''(x)\, dx.$$

Let $dw = u''(x)dx$ and $v(x) = u(x)$. We see that $w(x) = u'(x)$ and $d\Omega = u'(x)dx$. Then integration by parts tell us:

$$\int_a^b u(x)\, u''(x)\, dx = u\, u'\big|_a^b - \int_a^b \left[u'(x)\right]^2\, dx.$$

Here, the surface integral just becomes evaluation at the end points (that's the surface of a one dimensional line) and $\|\operatorname{grad}(u)\|^2$ is just $[u'(x)]^2$.

THEOREM 5.46. *Let $\Omega \subseteq \mathbb{R}^3$ and suppose that $u(\mathbf{x})$ solves the Laplace equation with boundary value $f(\mathbf{x})$ on $S = \partial\Omega$. Then $u(\mathbf{x})$ is the unique solution.*

PROOF. We proceed by contradiction. Suppose u and v are two such solutions. Then $w = u - v$ satisfies the PDE:

$$\begin{aligned} &\Delta w = 0 \\ &w(\mathbf{x}) = 0 \quad \mathbf{x} \in S. \end{aligned}$$

Then we can compute:

$$\int_\Omega w\, \Delta w\, d\Omega = 0,$$

since $\Delta w = 0$. Apply Green's first identity to see:

$$\int_\Omega \|\mathrm{grad}(w)\|^2\, d\Omega = \int_{\partial\Omega} [w\,\mathrm{grad}(w)] \cdot \mathbf{n}\, dS$$

But $w = 0$ on the boundary. Therefore, the right-hand-side must be zero, implying that:

$$\int_\Omega \|\mathrm{grad}(w)\|^2\, d\Omega = 0.$$

We know $\|\mathrm{grad}(w)\|^2 \geq 0$ everywhere and therefore (since the integral is zero) we can conclude that $w(\mathbf{x}) = C$, a constant in Ω (and on the boundary). But on the boundary $w(\mathbf{x}) = 0$, therefore $C = 0$ and thus $w(\mathbf{x})$ is zero everywhere in and on Ω. Thus we conclude $u(x) = v(x)$ in Ω and thus the solution to the Laplace equation is unique. □

Remark 5.47. We have now proved that the Laplace equation is well-posed; i.e., solutions vary continuously with the input parameters and are unique. We note that these results generalize to higher (and lower) dimensional versions of the Laplace equation – with slightly more complicated notation.

7. A Brief Result on Energy Methods

Remark 5.48. As we've hinted at before, non-linear PDE's are very difficult to solve. Some, however, can be attacked using general minimization principles. The motivation for this is an interesting property of the Laplace equation. Namely, that solutions minimize a certain *energy function*. Before proceeding, we need a lemma.

THEOREM 5.49 (Green's Second Identity). *Let $\Omega \subseteq \mathbb{R}^3$ and let $S = \partial\Omega$ and let $u, v : \mathbb{R}^3 \to \mathbb{R}$ be two scalar functions. Then:*

$$\int_\Omega u\, \Delta v\, d\Omega = \int_\Omega v\, \Delta u\, d\Omega + \int_S [u\,\mathrm{grad}(v) - v\,\mathrm{grad}(u)] \cdot \mathbf{n}\, dS. \tag{5.24}$$

PROOF. Let $\boldsymbol{\varphi} = \mathrm{grad}(u)$. Reason as before with the Divergence theorem we conclude:

$$\int_\Omega v\,\mathrm{div}(\boldsymbol{\varphi})\, d\Omega + \int_\Omega \boldsymbol{\varphi} \cdot \mathrm{grad}(v)\, d\Omega = \int_{\partial\Omega} (v\,\boldsymbol{\varphi}) \cdot \mathbf{n}\, dS,$$

or

$$\int_\Omega v\, \Delta u\, d\Omega + \int_\Omega \mathrm{grad}(u) \cdot \mathrm{grad}(v)\, d\Omega = \int_{\partial\Omega} [v\,\mathrm{grad}(u)] \cdot \mathbf{n}\, dS, \tag{5.25}$$

We can just as easily reverse the argument making $\boldsymbol{\varphi} = \text{grad}(v)$ to obtain:

$$\int_\Omega u\,\Delta v\,d\Omega + \int_\Omega \text{grad}(u)\cdot\text{grad}(v)\,d\Omega = \int_{\partial\Omega} [u\,\text{grad}(v)]\cdot\mathbf{n}\,dS, \tag{5.26}$$

Subtract Eq. (5.25) from Eq. (5.25) to obtain:

$$\int_\Omega u\,\Delta v\,d\Omega - \int_\Omega v\,\Delta u\,d\Omega = \int_{\partial\Omega} [u\,\text{grad}(v) - v\,\text{grad}(u)]\cdot\mathbf{n}\,dS.$$

Rearranging terms yields Eq. (5.24). □

Remark 5.50. This identify is also an integration by parts formula for the Laplacian and it is very useful in proving both the uniqueness of the solution to the Laplace equation and for proving that the Laplacian is a "self-adjoint" operator – which will be covered when we discuss higher dimensional PDE's.

THEOREM 5.51. *Let $\Omega \subseteq \mathbb{R}^3$ and let $u : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function. Define:*

$$E(u) = \int_\Omega \|\text{grad}(u)\|^2\,d\Omega$$

If u solves the Laplace equation:

$$\Delta u = 0$$
$$u(\mathbf{x}) = f(\mathbf{x}) \quad \mathbf{x} \in S,$$

then among all other functions w satisfying $w(\mathbf{x}) = f(\mathbf{x})$ on S we have $E(u) \leq E(w)$. That is, solutions to the Laplace equation minimize the energy functional E.

PROOF. Let $w = u+v$ where v solves the Laplace equation with zero boundary condition; that is $\Delta v = 0$ and $v(\mathbf{x}) = 0$ for $\mathbf{x} \in S$. Then:

$$E(w) = E(u+v) = \int_\Omega \|\text{grad}(u+v)\|^2\,d\Omega = \\ \int_\Omega \|\text{grad}(u)\|^2\,d\Omega + \int_\Omega \|\text{grad}(v)\|^2\,d\Omega + 2\int_\Omega \text{grad}(u)\cdot\text{grad}(v)\,d\Omega$$

Thus:

$$E(w) = E(u) + E(v) + 2\int_\Omega \text{grad}(u)\cdot\text{grad}(v)\,d\Omega.$$

Apply Eq. (5.25) from the proof of the second Green's Identity and observe that

$$\int_\Omega \text{grad}(u)\cdot\text{grad}(v)\,d\Omega = \int_{\partial\Omega} [v\,\text{grad}(u)]\cdot\mathbf{n}\,dS - \int_\Omega v\,\Delta u\,d\Omega.$$

We know $\Delta u = 0$ and for $\mathbf{x} \in S$ we assumed $v(\mathbf{x}) = 0$. Therefore, the right-hand-side must be zero and we conclude that:

$$E(w) = E(u) + E(v),$$

which implies $E(u) \leq E(w)$. □

Remark 5.52. Clearly we do not require this information to solve the Laplace equation. However similar minimization arguments drawing on *the calculus of variations* can be applied to analyze non-linear PDE's. See [Eva15] (Chapter 8).

CHAPTER 6

Non-homogeneous Problems and Sturm-Liouville Theory

1. Simple Inhomogeneous Boundary Conditions

Remark 6.1. We begin by considering the heat equation with non-homogeneous Dirichlet boundary value conditions, using information from the Laplace equation, which we just finished studying.

Derivation 6.2. Consider the heat equation with non–homogeneous Dirichlet boundary value conditions:

$$\begin{aligned}
&u_t = ku_{xx}\\
&u(x, 0) = f(x)\\
&u(0, t) = a\\
&u(L, t) = b.
\end{aligned}$$

In the ordinary heat equation with homogeneous Boundary or Neumann boundary conditions, we know that in the long-run the solution approaches a constant value from Proposition 2.31. This constant solution is the solution to the stationary (one-dimensional) Laplace equation:

$$\begin{aligned}
&u_{xx} = 0\\
&u(0) = u(L) = 0.
\end{aligned}$$

Let us assume that in the long-run our non-homogeneous problem will approach a solution to the problem:

$$\begin{aligned}
&u_{xx} = 0\\
&u(0) = a\\
&u(L) = b
\end{aligned}$$

This problem has a straightforward solution:

$$u(x) = \left(\frac{b-a}{L}\right)x + a$$

The goal is now to have a solution $u(x, t)$ with the property that:

$$\lim_{t\to\infty} u(x, t) = \left(\frac{b-a}{L}\right)x + a.$$

To accomplish this, let:

$$g(x) = f(x) - \left[\left(\frac{b-a}{L}\right)x + a\right].$$

If we now solve the problem:

$$\begin{aligned}
&\upsilon_t = k\upsilon_{xx}\\
&u(x,0) = g(x)\\
&u(0,t) = 0\\
&u(L,t) = 0.
\end{aligned}$$

and define:

$$u(x,t) = a + \left(\frac{b-a}{L}\right)x + \upsilon(x,t),$$

then we know:

$$\lim_{t\to\infty} \upsilon(x,t) = 0.$$

Therefore:

$$u(x,0) = a + \left(\frac{b-a}{L}\right)x + \upsilon(x,0) =$$

$$a + \left(\frac{b-a}{L}\right)x + g(x) = a + \left(\frac{b-a}{L}\right)x + f(x) - \left[a + \left(\frac{b-a}{L}\right)x\right] = f(x).$$

We also know that:

$$\begin{aligned}
&u(0,t) = a + \upsilon(0,t) = a\\
&u(L,t) = b + \upsilon(0,t) = b.
\end{aligned}$$

Finally, we compute:

$$u_t = \upsilon_t = k\upsilon_{xx} = k\partial_{xx}\left[a + \left(\frac{b-a}{L}\right)x + \upsilon\right] = ku_{xx}.$$

Therefore $u(x,t)$ solves the heat equation with the non-homogeneous Dirichlet boundary conditions.

2. Motivation for Generalization

Remark 6.3. In this chapter, we are going to further generalize the kind of problems we can study. So far we have found that every PDE we've encountered can be solved using separation of variables. One of the resulting equations always seems to satisfy the requirements of Lemma 2.15, Lemma 2.18, Lemma 2.21. Even the nonhomogeneous Dirichlet BC's we considered in the first section could be put into this form. What if this is not the case.

Exercise 41. Consider a temperature distribution on a circular plate (as in the Laplace equation) $u(r,\theta,t)$ does not depend on θ; i.e., it exhibits circular symmetry. Show that:

$$\Delta u = \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right). \tag{6.1}$$

[Hint: Use Proposition 5.13 and remember u *does not* depend on $\partial_\theta u = 0$. Show that Eq. (6.1) is the same a the polar Laplacian in that case.]

Example 6.4 (Motivating Problem 1)**.** Consider the heat equation on a disk under the assumption of circular symmetry. From Eq. (6.1) we have:

$$u_t = k\Delta u = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

If we assume that $u(r,t) = v(t)w(r)$ then we have:

$$v'(t)w(r) = \frac{k}{r}\frac{d}{dr}\left(r\frac{dw}{dr}v(t)\right).$$

Simplifying we have:

$$\frac{1}{k}\frac{v'}{v} = \frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)}{w} = -\lambda.$$

As before the separates into a time and a space ODE yielding:

$$v' = -k\lambda v$$
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\lambda w$$

We can simplify the space ODE as:

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -r\lambda w.$$

For the sake of argument, let $K(r) = \sigma(r) = r$ (the identify function). This ODE has form:

$$\frac{d}{dr}\left(K(r)\frac{dw}{dr}\right) = -\sigma(r)\lambda w, \tag{6.2}$$

and requires different analytic methods than the ones we have used so far.

Example 6.5 (Motivating Problem 2)**.** Recall from Eq. (2.3) in the derivation of the heat equation we had obtained:

$$\frac{\partial u}{\partial t} = -\frac{1}{c_0\rho(x)}\frac{\partial \varphi}{\partial x} + q(x,t), \tag{6.3}$$

except in Eq. (2.3) we assumed $\rho(x)$ was a constant ρ_0. We now remove that assumption and define:

$$h(x) = \frac{1}{c_0\rho(x)}. \tag{6.4}$$

Further, assume that the diffusion coefficient K in Fick's law (see Eq. (2.4)) is non-constant so that:

$$\varphi = -K(x)\frac{\partial u}{\partial x}. \tag{6.5}$$

Substituting Eq. (6.5) into Eq. (6.3) and using Eq. (6.4) we have the (more complex heat equation)

$$\frac{\partial u}{\partial t} = h(x)\frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right) + q(x,t) \tag{6.6}$$

Now consider the non-homogeneous heat equation:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = h(x)\frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right) + q(x,t)\\
&\alpha u(0) + \beta u_x(0) = a\\
&\alpha u(0) + \beta u_x(0) = b\\
&u(x,0) = f(x).
\end{aligned}
\tag{6.7}
$$

Here, α and β are real numbers. This problem has non-homogeneous diffusivity (heat), an internal heat source $q(x,t)$ (making the problem non-homogeneous) and a new kind of boundary condition (see Definition 6.6) that is also non-homogeneous. This problem will not readily yield to separation of variables in so simple a way as our previous problems did.

Definition 6.6 (Robin Boundary Condition)**.** Given a PDE defined by a linear operator L on a region Ω, a Robin boundary condition is of the form:

$$
\alpha u(\mathbf{x}) + \beta\nabla u(x)\cdot\mathbf{n} = a \quad \forall\mathbf{x}\in\partial\Omega,
$$

where a may (in fact) be a function.

Remark 6.7. Robin boundary conditions are the full generalization of both Neumann and Dirichlet boundary conditions. They are the fourth (and last) type of boundary condition we consider and are more difficult to deal with that either Neumann or Dirichlet boundary conditions. From a physical perspective, in one dimension, we can think of a Robin BC as indicating that heat is radiating away from the boundary in an analog to Newton's law of cooling. In the case of a vibrating string, a Robin BC indicates a restoring force at the boundary.

Derivation 6.8. For the sake of argument, let us make Eq. (6.7) slightly easier and assume that $q(x,t) = \gamma u(x,t)$. That is, the internal heat generation is proportional to the temperature itself. The parameter γ may be a function of x but not of t for now.

We now attempt to use separation of variables on Eq. (6.7). Let $u(x,t) = v(t)w(x)$. Expanding the partial differential equation we have:

$$
v'(t)w(x) = v(t)h(x)\frac{d}{dx}\left(K(x)\frac{dw}{dx}\right) + \gamma v(t)w(x)
$$

Dividing through we see:

$$
\frac{v'}{v} = \frac{h(x)\frac{d}{dx}\left(K(x)\frac{dw}{dx}\right) + \gamma w(x)}{w} = -\lambda.
$$

As before we obtain time and a space ODE's:

$$
v' + \lambda v = 0 \tag{6.8}
$$

$$
h(x)\frac{d}{dx}\left(K(x)\frac{dw}{dx}\right) + \gamma w(x) + \lambda w(x) = 0 \tag{6.9}
$$

Consider the space equation alone. Dividing through by $h(x)$ we can write this as:

$$\frac{d}{dx}\left(K(x)\frac{dw}{dx}\right) + \frac{\gamma}{h(x)}w(x) + \frac{\lambda}{h(x)}w(x) = \\ \frac{d}{dx}\left(K(x)\frac{dw}{dx}\right) + \gamma c_0\rho(x)w(x) + c_0\rho(x)\lambda w(x).$$

Clearing the debris away this has form:

$$\frac{d}{dx}\left(K(x)\frac{dw}{dx}\right) + q(x)w = -\lambda\sigma(x)w. \tag{6.10}$$

Remark 6.9. We note that the two equations Eqs. (6.2) and (6.10) have a similar form. Of course in Eq. (6.2), $q(r) = 0$. You will note, further than when $K(x) = \sigma(x) = 1$ and $q(x) = 0$, we even recover the problem from Lemmas 2.15, 2.18 and 2.21:

$$w'' = -\lambda w.$$

The goal of the rest of this chapter is to study these kinds of ODE's, since they seem to emerge naturally in separation of variables in PDE's. Not only that, but at least one of these problems has interesting solutions that form a basis for a function space, allowing us to deal nicely with initial conditions.

3. Sturm-Lioville Differential Equations

Definition 6.10. Let $p(x)$, $q(x)$ and $\sigma(x)$ be functions of the independent variable x. The function $\sigma(x)$ is called the *weight function*. Let $w(x)$ be an unknown function of x satisfying the boundary value problem:

$$\begin{cases} \frac{d}{dx}\left(p(x)\frac{dw}{dx}\right) + q(x)w = -\lambda\sigma(x)w. \\ \alpha_1 w(a) + \beta_1 w'(a) = 0 \\ \alpha_2 w(b) + \beta_2 w'(b) = 0 \end{cases} \tag{6.11}$$

on the interval $[a, b]$. Finding w values of λ for which $w(x)$ is non-trivial is called a *regular Sturm-Lioville Eigenvalue Problem* (SLEP) and the ODE is called a *Sturm-Lioville differential equation*. A value λ is an *eigenvalue* just in case there is an *eigenfunction* solution $w(x)$ to the SLEP for that value λ.

Remark 6.11. The boundary conditions in SLEP arise from Robin boundary conditions. In general, we will have $a = 0$ and $b = L$. Any values will suffice as the results are translation invariant.

Remark 6.12. In what follows, we're going to (more or less) follow Haberman [**Hab03**] but with some input from Logan [**Log14**]. Neither are particularly general. For a more general approach see [**Olv14**] (Chapter 9) – which seems very complicated on a first read but is not that bad. If you're going to use [**Olv14**], it helps to have done a course in Linear Algebra where you cover the spectral theorem of Hermitian matrices (or real-symmetric matrices). A good portion of this material is just a generalization of that. In fact, the proofs are basically the same.

Remark 6.13. We state but do not prove the Sturm-Liouville theorem in its entirety. We will, however, proof pieces of it. Before that, we must generalize our notion of inner product.

Definition 6.14 (Weighted inner product)**.** Let $\sigma : [a, b] \to \mathbb{R}$ be a weight function. If $f, g : [a, b] \to \mathbb{R}$ then:

$$\langle f, g\rangle_\sigma = \int_a^b f(x)g(x)\sigma(x)\, dx \tag{6.12}$$

Remark 6.15. When $\sigma(x) = 1$, this is just the standard inner product from Definition 3.15 and we omit the subscript 1.

THEOREM 6.16 (Sturm-Lioville Theorem). *Consider a SLEP*

$$\begin{cases} \dfrac{d}{dx}\left(p(x)\dfrac{dw}{dx}\right) + q(x)w = -\lambda\sigma(x)w. \\ \alpha_1 w(a) + \beta_1 w'(a) = 0 \\ \alpha_2 w(b) + \beta_2 w'(b) = 0. \end{cases}$$

Then:

1. *There are a countable set of unique real eigenvalues* $\lambda_1 < \lambda_2 < \cdots$ *with* $\lim_{n\to\infty} \lambda_n = +\infty$.
2. *For each eigenvalue* λ_n *there is a unique (up to a constant multiple) eigenfunction* $w_n(x)$.
3. *If* $n \neq m$, *then* $\langle w_n, w_m\rangle_\sigma = 0$. *That is, the eigenfunctions are orthogonal.*
4. *The eigenfunctions are complete in the sense that for* $x \in [a, b]$, *if* $f(x)$ *is piecewise* C^2, *then:*

$$f(x) \sim \sum_{n=1}^{\infty} a_n w_n(x) = \hat{f}(x), \tag{6.13}$$

with:

$$a_n = \frac{\langle w_n, f\rangle_\sigma}{\langle w_n, w_n\rangle_\sigma}. \tag{6.14}$$

Further if $f(x)$ *has a jump discontinuity at* x, *then the series converges to:*

$$\hat{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right].$$

□

Remark 6.17. Eq. (6.13) is called a *Generalized Fourier Series.*

Remark 6.18. Proving Theorem 6.16 is outside the scope of the course for the same reason we did not prove convergence of the Fourier series in Chapter 3. However, when $\sigma(x) = 1$, [**AG08**] has a complete and reasonably readable proof of this result.

4. Some Theoretical Results

Remark 6.19. Let L be the linear operator define as:

$$L(w) = \frac{d}{dx}\left(p(x)\frac{dw}{dx}\right) + q(x)w.$$

That is, $L(w)$ is just the right-hand-side of the SLEP ODE. We now analyze this linear operator to prove (2) and (3) of Theorem 6.16. Note, that $\sigma(x)$ only appears in defining the inner product. Therefore, most books (e.g., [**Log14**]) just set it equal to 1.

Remark 6.20. For brevity we're going to suppress the independent variable and write (for example) p for $p(x)$ and q for $q(x)$.

Derivation 6.21 (Lagrange's Identity)**.** Consider two C^1 functions u and v and compute:

$$uL(v) - vL(u) = u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + uqv - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - vqu = u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right).$$

Now cleverly add zero:

$$uL(v) - vL(u) = u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + p\frac{dv}{dx}\frac{du}{dx} - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - p\frac{du}{dx}\frac{dv}{dx}$$

Now working the product rule backward we have:

$$uL(v) - vL(u) = \frac{d}{dx}\left(pu\frac{dv}{dx}\right) - \frac{d}{dx}\left(pv\frac{du}{dx}\right) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]. \tag{6.15}$$

The expression:

$$W(x) = u\frac{dv}{dx} - v\frac{du}{dx} \tag{6.16}$$

is called the *Wronskian*. We have proved a lemma.

Lemma 6.22. *Let $u(x)$ and $v(x)$ be two C^1 functions. Then:*

$$uL(v) - vL(u) = \frac{d}{dx}\left[pW(x)\right],$$

where $W(x)$ is the Wronskian of $u(x)$ and $v(x)$. □

Corollary 6.23 (Green's Second Identity Redux)**.** *Suppose $u(x)$ and $v(x)$ be two C^1 functions on the interval $[a, b]$. Then:*

$$\int_a^b uL(v) - vL(u)\,dx = [pW(x)]|_a^b \tag{6.17}$$

□

Exercise 42. Prove the corollary by integrating Eq. (6.15) and applying the fundamental theorem of integral calcuus.

Exercise 43. Compare the corollary to Theorem 5.49 and show they are the same when $q(x) = 0$ and $p(x) = 1$. [Hint: Notice that the surface integral of a line simplifies to evaluating the integrand at the end points. The only difference is the operator being used.]

Lemma 6.24. *Suppose u and v satisfy Sturm-Liouville ODE and in particular the Robin boundary conditions. Then:*

$$\int_a^b uL(v) - vL(u)\,dx = [pW(x)]|_a^b = 0.$$

PROOF. Evaluate:

$$u\frac{dv}{dx} - v\frac{du}{dx}\bigg|_a^b = u(b)v'(b) - u(a)v'(a) + v(a)u'(a) - v(b)u'(b).$$

Without loss of generality, assume $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ and that $\beta \neq 0$. From the boundary conditions we know:

$$\frac{v'(a)}{v(a)} = \frac{u'(a)}{u(a)} = -\frac{\alpha}{\beta}.$$

Cross-multiplying we have:

$$v'(a)u(a) = u'(a)v(a) = -\frac{\alpha}{\beta}u(a)v(a). \tag{6.18}$$

A similar result holds for $x = b$ and thus:

$$u\frac{dv}{dx} - v\frac{du}{dx}\bigg|_a^b = -\frac{\alpha}{\beta}u(b)v(b) + \frac{\alpha}{\beta}u(a)v(a) - \frac{\alpha}{\beta}u(a)v(a) + \frac{\alpha}{\beta}u(b)v(b) = 0.$$

A similar argument holds if $\alpha \neq 0$. □

Remark 6.25 (Self-Adjointness)**.** Notice that:

$$\int_a^b uL(v) - vL(u)\,dx = \langle u, L(v)\rangle - \langle L(u), v\rangle = 0.$$

Thus, the previous lemma asserts assuming the Robin boundary conditions, the Sturm-Liouville operator satisfies:

$$\langle L(u), v\rangle = \langle u, L(v)\rangle. \tag{6.19}$$

An arbitrary operator (any dimension) operating on an appropriate function space with an appropriately defined inner product that satisfies Eq. (6.19) is called *self-adjoint*. Unfortunately to really understand what that means, we should define an adjoint to an operator. (Effectively, it's like a generalized transpose of a matrix.) For those interested in this topic, see Chapter 9 of [**Olv14**].

Proposition 6.26 (Distinct Eigenfunctions are Orthogonal)**.** *If the eigenfunction/eigenvalue pairs (w_m, λ_m) and (w_n, λ_n) solve the SLEP with Robin boundary conditions and $\lambda_m \neq \lambda_n$, then w_m and w_n are orthogonal. That is:*

$$\langle w_m, w_n\rangle_\sigma = \int_a^b \sigma(x)w_m(x)w_n(x)\,dx = 0$$

PROOF. By assumption we must have:

$$L(w_m) = \sigma\lambda_m w_m \tag{6.20}$$

$$L(w_n) = \sigma\lambda_n w_n, \tag{6.21}$$

because these eigenfunction/eigenvalue pairs satisfy the SLEP. Apply Lemma 6.24 to see:

$$\int_a^b w_m L(w_n) - w_n L(w_m)\,dx = 0 = \int_a^b w_m\sigma\lambda_n w_n - w_n\sigma\lambda_m w_m\,dx.$$

Factor the right-hand-side to see:

$$(\lambda_n - \lambda_m)\int_a^b \sigma(x)w_m(x)w_n(x)\,dx = 0. \tag{6.22}$$

We assumed $\lambda_n \neq \lambda_m$. Therefore we conclude $\langle w_m, w_n \rangle_\sigma = 0$. $\square$

Proposition 6.27 (Unique Eigenfunction). *Suppose (u, λ) is an eigenfunction/eigenvalue pair for the SLEP. If (v, λ) is a second eigenfunction with the same eigenvalue, then there is a constant $c \in \mathbb{R}$ so that:*

$$v = cu.$$

PROOF. Apply Lemma 6.24 to see:

$$\int_a^b uL(v) - vL(u)\,dx = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right] = 0.$$

Then there is a constant c so that:

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) = C.$$

Evaluate at $x = a$ and $x = b$ and use Eq. (6.18) to see:

$$p\left(u(a)v'(a) - v(a)u'(a)\right) = 0 = \left(u(b)v'(b) - v(b)u'(b)\right).$$

Therefore $C = 0$. Assuming $p(x) \neq 0$, we conclude:

$$u\frac{dv}{dx} - v\frac{du}{dx} = 0.$$

This is the numerator of the quotient rule, which means we conclude that:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = 0.$$

Therefore there is a $c \in \mathbb{R}$ so that:

$$\frac{v}{u} = c$$

and we conclude $v = cu$. $\square$

Proposition 6.28 (The eigenvalues of a SLEP are real.). *Suppose (w, λ) is an eigenfunction/eigenvalue pair for a SLEP. Then λ is real.*

PROOF. Proceed by contradiction. Suppose u is complex and λ is complex and consider their conjugates $\overline{w}$ and $\overline{\lambda}$. We assume $p(x)$, $q(x)$ and $\sigma(x)$ are all real. Therefore:

$$\overline{L(w) + \lambda\sigma w} = \overline{L(w)} + \overline{\lambda\sigma w} = L(\overline{w}) + \overline{\lambda}\sigma\overline{w} = 0.$$

Consider $x = a$. Then:

$$\overline{\alpha w(a) + \beta w'(a)} = \overline{w}(a) + \beta\overline{w}'(a) = 0.$$

A similar result holds at $x = b$ and $(\overline{w}, \overline{\lambda})$ is a second eigenfunction/eigenvalue pair. Applying Eq. (6.22) we have:

$$\left(\lambda - \overline{\lambda}\right)\int_a^b \sigma(x)w(x)\overline{w}(x)\,dx = \left(\lambda - \overline{\lambda}\right)\int_a^b \sigma(x)[w(x)]^2\,dx = 0.$$

We know that:

$$\int_a^b \sigma(x)[w(x)]^2\,dx > 0,$$

because $w(x) \neq 0$. Therefore $\lambda - \overline{\lambda} = 0$ or $\lambda = \overline{\lambda}$, which can only be true if λ is real. $\square$

5. Raleigh Quotient

Proposition 6.29 (Raleigh Quotient)**.** *Suppose that w solves a SLEP with corresponding eigenvalue λ. Then:*

$$\lambda = \frac{-pw\frac{dw}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{dw}{dx}\right)^2 - qw^2\right]\,dx}{\int_a^b w^2\sigma\,dx} \tag{6.23}$$

Proof. Multiply the SLEP by $w(x)$ to obtain:

$$wL(w) = \lambda\sigma w^2.$$

Integrating we see that:

$$\int_a^b wL(w)\,dx = -\lambda\int_a^b \sigma w^2\,dx.$$

Substituting we have:

$$\int_a^b wL(w)\,dx = \int_a^b w\frac{d}{dx}\left(p\frac{dw}{dx}\right) + qw^2\,dx =$$
$$\int_a^b w\frac{d}{dx}\left(p\frac{dw}{dx}\right)\,dx + \int_a^b qw^2\,dx = -\lambda\int_a^b \sigma w^2\,dx.$$

Use integration by parts on the first integral with $u(x) = w(x)$, $dv = (pw')'\,dx$ therefore $du = w'\,dx$ and $v = pw'$. Substituting we have:

$$\int_a^b w\frac{d}{dx}\left(p\frac{dw}{dx}\right)\,dx = wp\frac{dw}{dx}\Big|_a^b - \int_a^b p\left(\frac{dw}{dx}\right)^2\,dx$$

We know that:

$$\int_a^b \sigma w^2\,dx > 0,$$

therefore we can solve:

$$\lambda = -\frac{wp\frac{dw}{dx}\Big|_a^b - \int_a^b p\left(\frac{dw}{dx}\right)^2\,dx + \int_a^b qw^2\,dx}{\int_a^b \sigma w^2\,dx}.$$

Simplifying we obtain Eq. (6.23). □

Corollary 6.30. *Consider a SLEP with eigenfunction $w(x)$ and eigenvalue λ. If:*

$$p(x) \geq 0$$
$$q(x) \leq 0$$
$$-pw\frac{dw}{dx}\Big|_a^b \geq 0,$$

then $\lambda > 0$.

Example 6.31. Consider the SLEP with $p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$. And assume that $a = 0$, $b = L$, $\alpha = 1$, and $b\beta = 0$. The problem is:

$$w'' + \lambda w = 0$$
$$w(0) = w(L) = 0.$$

We have already proved in Lemma 2.15 that the eigenfunctions and eigenvalues are:

$$\begin{aligned}
w_n(x) &= \sin\left(\frac{n\pi x}{L}\right) \\
\lambda_n &= \left(\frac{n\pi}{L}\right)^2.
\end{aligned}$$

We can confirm this using the Raleigh quotient. We have:

$$w'(x) = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right).$$

Computing we see:

$$pw(x)w'(x)|_0^L = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0.$$

We can also compute:

$$\int_0^L \sigma w^2\,dx = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\,dx = \frac{L}{2}.$$

Likewise:

$$\int_0^L (w')^2\,dx = \left(\frac{n\pi}{L}\right)^2 \int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\,dx = \frac{n^2\pi^2}{2L}.$$

Computing the Raleigh Quotient we see:

$$\frac{\int_0^L (w')^2\,dx}{\int_0^L \sigma w^2\,dx} = \frac{n^2\pi^2}{2L}\frac{2}{L} = \frac{n^2\pi^2}{L^2},$$

as expected.

Remark 6.32. The real use of the Raleigh Quotient is in bounding the eigenvalues of the operator L. Information about the eigenvalues can be useful since these often provide physical information about the problem in question. We state but do not prove the following theorem.

THEOREM 6.33. *Consider the Raleigh Quotient as a mapping from a function w to a real value λ. That is, $R : w \mapsto \lambda$, where R is the Raleigh Quotient. Then among all functions u we have:*

$$R(u) \geq \lambda_1,$$

where λ_1 is the smallest eigenvalue of the SLEP. □

Remark 6.34. A mapping from a function to a scalar is usually called a *functional*. Proving Theorem 6.33 usually applies techniques from the Calculus of Variations. [**Hab03**] has a proof that assumes integration and summation can be exchanged. This is very readable but also non-standard since the assumption requires certain convergence guarantees that are not justified. (Otherwise, it's a perfectly nice proof.)

6. Singular SLEP's

Definition 6.35. Consider a SLEP. If $p(x) = 0$ for some value $x_0 = [a, b]$ or one or more of a and b are infinite, then the problem is called *singular*. Otherwise it is *regular*.

Example 6.36 (Bessel's Differential Equation)**.** The differential equation:

$$\frac{d}{dx}\left(x\frac{dw}{dx}\right) + \left(x - \frac{m^2}{x}\right) w = 0, \tag{6.24}$$

is a singular SLEP when $x = 0$. This equation can be re-written in its more standard form:

$$x^2\frac{d^2w}{dx^2} + x\frac{dw}{dx} + (x^2 - m^2) = 0. \tag{6.25}$$

Here, m^2 is going to play the role of λ. Like all second order ODE's, this equation has two solutions. From the Sturm-Liouville theory we have encountered thus far, we know these two solutions must be composed of orthogonal functions that can be used as a basis to construct other functions.

Solving these equations requires the use of series solutions methods, which (unfortunately) we do not have time to discuss. This approach, called the Frobenius method is covered in [**Inc12**]. The interested reader can also consult [**Olv14**] or [**Asm16**].

We generally denote these two families of solutions as:

- Bessel Functions of the First Kind J_m, which are decaying oscillations with an infinite number of zeros and
- Bessel Functions of the Second Kind Y_m, which are unbounded at the origin.

The Bessel function of the first kind $J_m(x)$ can be expressed in the series:

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m+1)!}\left(\frac{x}{2}\right)^{2n+m},$$

for integer m. It can also be extended to non-integer m using the Gamma function – which generalizes the factorial function. The formula for $Y_m(x)$ is more complex – especially for integer values m. We will not use this kind of Bessel function, so we can safely ignore its exact formulation in these notes. The two kinds of solutions are illustrated in Fig. 6.1. However, there are other ways to construct the solution families. The general solution to Bessel's equation then is:

$$w(x) = c_1 J_m(x) + c_2 Y_m(x),$$

with the constants determined by the boundary conditions.

Note that the functions Y_m are unbounded at $x = 0$, which is the singular point, while J_m are well behaved. We have already seen a situation like this when we studied the Laplace equation on a disk, where one class of solutions to an ODE was bounded at 0 and the other class was not.

7. The Non-Uniform Heat Equation

Remark 6.37. We now return to the non-uniform heat equation and provide a (somewhat unsatisfying) analysis.

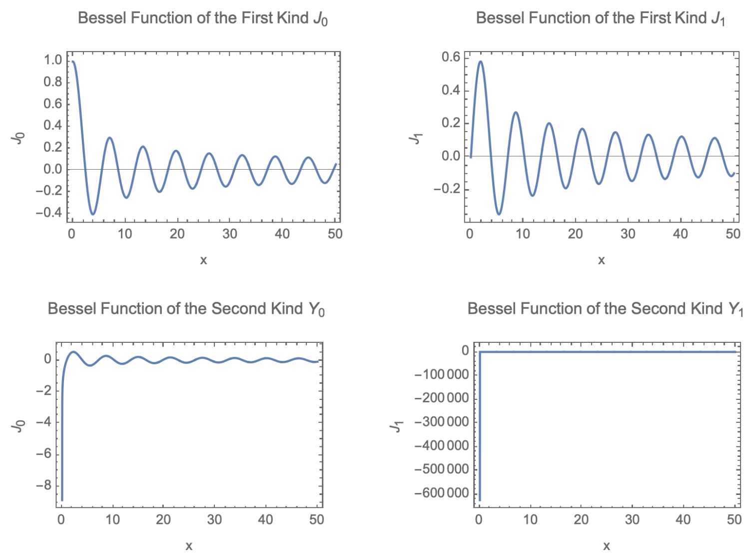

FIGURE 6.1. Examples of the Bessel Functions. The function Y_m is always unbounded at 0 for all m.

Derivation 6.38. Consider the non-homogeneous heat equation from Eq. (6.7) but for simplicity, assume $h(x) = 1$, $q(x) = 0$ and let us introduce mixed boundary conditions to obtain:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right) \\ &u(0) = 0 \\ &u_x(L) = 0 \\ &u(x,0) = f(x). \end{aligned} \tag{6.26}$$

Let $u(x,t) = v(t)w(x)$. Separating variables we obtain the time ODE:

$$v' = -\lambda v$$

and the space ODE:

$$\begin{aligned} &\frac{d}{dx}\left(K(x)\frac{dw}{dx}\right) + \lambda w = 0 \\ &w(0) = 0 \\ &w'(L) = 0. \end{aligned}$$

The space ODE is a SLEP. Therefore, we have a family of orthogonal eigenfunction solutions $w_n(x)$ with corresponding eigenvalues λ_n. (How we could actually find this family of solutions is completely dependent on what $K(x)$ looks like.)

We assume that $K(x) \geq 0$ (this is physical) and:

$$K(x)w\frac{dw}{dx}\bigg|_0^L = 0.$$

Therefore from Corollary 6.30 we have $\lambda_n > 0$. Therefore we can write:

$$u(x,t) = \sum_{n=1}^{\infty} a_n w_n(x) e^{-\lambda_n t}.$$

We require:

$$f(x) = \sum_{n=1}^{\infty} a_n w_n(x).$$

We can derive the coefficients as:

$$a_n = \frac{\int_0^L f(x) w_n(x)\, dx}{\int_0^L \left[w_n(x)\right]^2\, dx} = \frac{\langle f, w_n\rangle}{\langle w_n, w_n\rangle} = \frac{\langle f, w_n\rangle}{\|w\|^2}.$$

We know that $\lambda_n > 0$ for all n. We can take the limit:

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \sum_{n=1}^{\infty} a_n w_n(x) e^{-\lambda_n t} = 0.$$

This is expected from a 1D heat equation where one end of the rod is insulated and the other end is fixed at temperature 0. The heat will "drain out" of the non-insulated end.

8. Robin Boundary Conditions

Example 6.39. Consider the following heat equation with Robin boundary conditions:

$$\begin{aligned}
&u_t = u_{xx}\\
&u(x,0) = f(x)\\
&u_x(0,t) = 0\\
&u(1,t) + u_x(1,t) = 0
\end{aligned}$$

Physically this corresponds to rod with an insulated end and another end with a radiating boundary. Notice the heat constant $k = 1$ in this problem. Let $u(x,t) = v(t)w(x)$. We have the time ODE:

$$v' = -\lambda v$$

and the space ODE:

$$\begin{aligned}
&w'' = -\lambda w\\
&w'(0) = 0.\\
&w(1) + w'(1) = 0
\end{aligned}$$

The space ODE is a SLEP, which we can solve. We know that for $\lambda > 0$ the solution is of the form:

$$w(x) = C_1 \sin\left(\sqrt{\lambda} x\right) + C_2 \cos\left(\sqrt{\lambda} x\right).$$

Since $w'(0) = 0$, we see that:

$$\sqrt{\lambda} C_1 = 0,$$

which implies that $C_1 = 0$. Using the second boundary condition we have:

$$C_2 \cos\left(\sqrt{\lambda}\right) - C_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right).$$

This implies that:

$$\cos\left(\sqrt{\lambda}\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) = 0$$

Therefore, $\sqrt{\lambda}$ must satisfy the relation:

$$\cot\left(\sqrt{\lambda}\right) = \sqrt{\lambda}.$$

We can see this relationship yields an infinite set of eigenvalues as expected (see Fig. 6.2). Unlike the ordinary heat equation, we would need to solve for these eigenvalues numerically. Denote the eigenvalues (that can be found numerically) as $\lambda_1, \lambda_2, \dots$. We have deduced that:

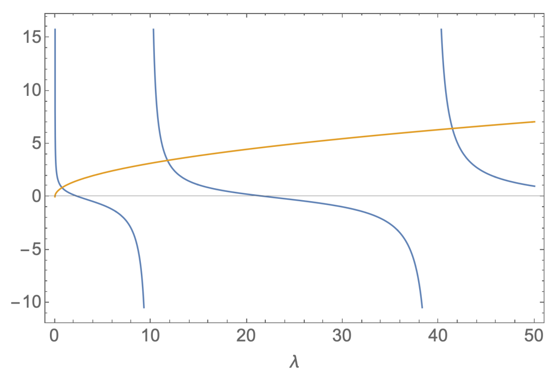

FIGURE 6.2. The eigenvalues of the space ODE (SLEP) can be identified numerically, but not in closed form.

$$w_n(x) = a_n \cos\left(\sqrt{\lambda_n} x\right),$$

where the eigenfunctions are $\cos\left(\sqrt{\lambda_n} x\right)$, which form a basis for an appropriate function space. As usual, we can now deduce that:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left[-\lambda_n t\right] \cos\left(\sqrt{\lambda_n} x\right).$$

The coefficients a_n can be computed using Eq. (6.14) as:

$$a_n = \frac{\int_0^1 f(x)\cos\left(\sqrt{\lambda_n} x\right)\,dx}{\int_0^1 \cos^2\left(\sqrt{\lambda_n} x\right)\,dx}.$$

9. The Forced Heat Equation

Derivation 6.40. Consider the non-homogeneous PDE:

$$\begin{aligned}
&u_t = k u_{xx} + q(x,t) \\
&u(x,0) = f(x) \\
&\alpha_1 u(0,t) + \beta_1 u_x(0,t) 0 \\
&\alpha_2 u(L,t) + \beta_2 u_x(L,t) = 0
\end{aligned}$$

First we note we could assume a non-homogeneous material as we did in Derivation 6.38 but we're going to assume a uniform material so that we don't get lost in the details.

Consider the homogeneous PDE:

$$u_t = ku_{xx}$$
$$u(x,0) = f(x)$$
$$\alpha_1 u(0,t) + \beta_1 u_x(0,t) = 0$$
$$\alpha_2 u(L,t) + \beta_2 u_x(L,t) = 0$$

We know this PDE can be solved using separation of variables. The result is a solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} \exp[-k\lambda_n t] w_n(x),$$

where $w_n(x)$ and λ_n are eigenfunctions and eigenvalues of the ODE:

$$w'' + \lambda w = 0$$
$$\alpha_1 w(0) + \beta_1 w'(0) = 0$$
$$\alpha_2 w(L) + \beta_2 w'(L) = 0$$

From Theorem 6.16, we know we can construct a series representation for functions in x using $w_n(x)$. Therefore express $q(x,t)$ as:

$$q(x,t) = \sum_{n=1}^{\infty} a_n(t) w_n(x).$$

Notice the coefficients are now in terms of t but can still be expressed as:

$$a_n(t) = \frac{\int_0^L q(x,t) w_n(x)\, dx}{\int_0^L w_n(x)^2\, dx}.$$

Now consider each n separately and suppose there is a $v_n(t)$ corresponding to λ_n and $a_n(t)w_n(t)$. That is, let $u_n(x,t) = v_n(t)w_n(x)$. Then the original PDE becomes:

$$v_n'(t)w_n(x) = kv_n(t)w_n''(x) + a_n(t)w_n(x).$$

Separating the variables we see:

$$\frac{1}{k}\frac{v_n' - a_n(t)}{v_n} = \frac{w_n''}{w_n} = -\lambda_n.$$

We already know what $w_n(x)$ is, having used the spatial component of the homogeneous PDE. We now see that the time ODE becomes:

$$v_n'(t) + k\lambda_n v_n(t) - a_n(t) = 0.$$

Recall though, we have the initial condition $u(x,0) = f(x)$. This means that:

$$f(x) = \sum_{n=1}^{\infty} v_n(0) w_n(t).$$

Let us expand $f(x)$ as:

$$f(x) = \sum_{n=1}^{\infty} c_n w_n(x).$$

Then equating terms we have:

$$v_n(0) = c_n.$$

Thus we have an infinite family of time ODE's:

$$v_n'(t) + k\lambda_n v_n(t) - a_n(t) = 0.$$
$$v_n(0) = c_n.$$

These ODE's can be solved (by any means necessary) to obtain a complete solution to the PDE.

Example 6.41. Consider the forced heat equation:

$$u_t = u_{xx} + \sin(\pi x)$$
$$u(x,0) = 0$$
$$u(0,t) = u(L,t) = 0$$

The homogeneous boundary value problem when we separate variables is just:

$$w'' + \lambda w = 0$$
$$w(0) = w(1) = 0.$$

This has solution $w_n(x) = b_n \sin(n\pi x)$ with $\lambda = n^2\pi^2$ because $L = 1$. This makes it very easy to expand the forcing function $q(x) = \sin(\pi x)$. The expansion is simply itself; that is $a_1(t) = 1$ and $a_n(t) = 0$ for $n \neq 1$. The Fourier expansion of $f(x)$ in $\sin(n\pi x)$ is just 0 (i.e., $c_n = 0$ for all n). Therefore we have the time ODE's:

Case $n = 1$:

$$v_1'(t) + \pi^2 v_1(t) - 1 = 0.$$
$$v_1(0) = 0$$

Case $n \neq 1$:

$$v_n'(t) + n^2\pi^2 v_n(t) = 0.$$
$$v_n(0) = 0$$

When $n \neq 1$, we know $v_n(t) = 0$. When $n = 1$ the solution to the time ODE is[1]:

$$v_1(t) = \frac{e^{-\pi^2 t}\left(e^{\pi^2 t} - 1\right)}{\pi^2}.$$

Therefore, we have:

$$u(x,t) = \sum_{n=1}^{\infty} v_n(t) w_n(x) = v_1(t) w_1(x) = \left(\frac{e^{-\pi^2 t}\left(e^{\pi^2 t} - 1\right)}{\pi^2}\right)\sin(\pi x).$$

The temperature distribution in time is shown in Fig. 6.3.

Remark 6.42. This approach can be generalized and is generalized in Chapter 8 of [**Hab03**]. The essential ideas, however, are the same. Though there is a nice use of Green's Second Identity for SLEP (see Corollary 6.23).

[1] I used Mathematica here. You can separate variables in the ODE sense to solve this.

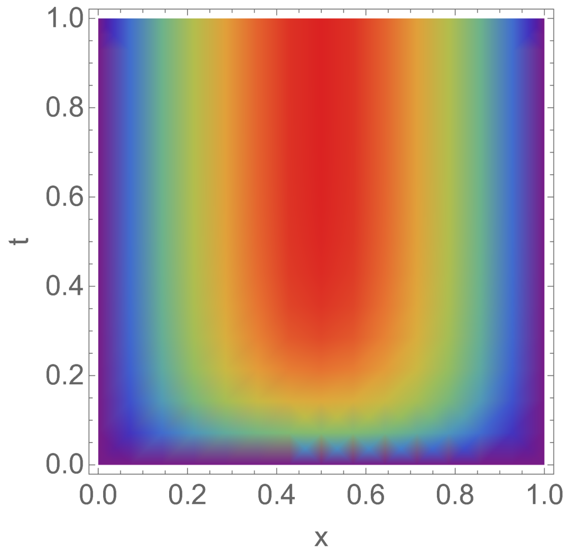

FIGURE 6.3. A solution to the forced heat equation shows that in the long run $u(x, t)$ approaches $q(x)$, as expected.

CHAPTER 7

Higher Dimensional PDE's

1. Planar Wave Equation

Remark 7.1. In this section we will analyze the planar wave equation using Dirichlet Boundary Conditions. The analysis for the planar heat equation is almost identical and will be left as an exercise.

Derivation 7.2. Consider the two dimensional problem in Cartesian coordinates:

$$\begin{aligned}
&u_{tt} = c^2\left(u_{xx} + u_{yy}\right) \\
&u(0, y, t) = u(L, y, t) = 0 \quad y \in [0, H] \\
&u(x, 0, t) = u(x, H, t) = 0 \quad x \in [0, L] \\
&u(x, y, 0) = f(x, y) \\
&u_t(x, y, 0) = g(x, y).
\end{aligned}$$

We will approach this problem just as before by assuming that we can write:

$$u(x, y, t) = v(t)w(x)p(y).$$

Then we have:

$$v''(t)w(x)p(y) = c^2\left[v(t)w''(x)p(y) + v(t)w(x)p''(y)\right].$$

Divide through by $c^2v(t)w(x)p(y)$ to obtain:

$$\frac{1}{c^2}\frac{v''}{v} = \frac{w''}{w} + \frac{p''}{p} = -\lambda.$$

Immediately, we see the time ODE is:

$$v'' + c^2\lambda v = 0.$$

The space PDE can be rewritten as:

$$\frac{w''}{w} = -\frac{p''}{p} - \lambda = -\mu,$$

where μ is another constant. We can write this because *again* the left-hand-side is only in terms of x and the right-hand-side is only in terms of y. We now have two space ODE's:

$$\begin{aligned}
&w'' + \mu w = 0, w(0) = w(L) = 0 \\
&p'' + (\lambda - \mu)p = 0, p(0) = p(H) = 0.
\end{aligned}$$

From Lemma 2.15, we know that:

$$w_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right)$$

and

$$\mu_n = \left(\frac{n\pi}{L}\right)^2.$$

By similar reasoning, we deduce:

$$\kappa_m = \lambda - \mu_n = \left(\frac{m\pi}{H}\right)^2$$

and therefore:

$$p_m(y) = k_m \sin\left(\frac{m\pi y}{H}\right).$$

We also deduce that:

$$\lambda_{mn} = \kappa_m + \mu_n = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2.$$

Now the solution to the time ODE is given by:

$$v_{mn}(t) = a_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right).$$

We now compute:

$$u_{mn}(x,y,t) = a_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) + \\ b_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

Thus:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) + \\ b_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

Now we have:

$$u(x,y,0) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right).$$

This is a double Fourier series with the basis functions:

$$\phi_{mn}(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right).$$

Generalizing from before, we can compute:

$$a_{mn} = \frac{\int_0^L\int_0^H f(x,y)\phi_{mn}(x,y)\,dy\,dx}{\int_0^L\int_0^H [\phi_{mn}(x,y)]^2\,dy\,dx}.$$

To compute the denominator, we have:

$$\int_0^L\int_0^H \sin^2\left(\frac{n\pi x}{L}\right)\sin^2\left(\frac{m\pi y}{H}\right)\,dy\,dx = \frac{LH}{4}.$$

Therefore we conclude:

$$a_{mn} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\,dy\,dx.$$

Differentiation term by term for $u_t(x, y, 0) = g(x, y)$ we see:

$$u_t(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} c\sqrt{\lambda_{mn}} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$

Let:

$$h(x, y) = \frac{1}{c\sqrt{\lambda_{mn}}} g(x, y).$$

Then we compute a double Fourier decomposition of $h(x, y)$ and see:

$$b_{mn} = \frac{4}{LH}\int_0^L \int_0^H h(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)\, dy\, dx = \\ \frac{4}{LHc\sqrt{\lambda_{mn}}}\int_0^L \int_0^H g(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)\, dy\, dx.$$

Remark 7.3. This derivation allows us to state the next proposition.

Proposition 7.4. *Assuming term-by-term differentiation is allowable, the solution to the wave equation with homogeneous Dirichlet boundary conditions in the plane:*

$$\begin{aligned}
&u_{tt} = c^2 (u_{xx} + u_{yy}) \\
&u(0, y, t) = u(L, y, t) = 0 \quad y \in [0, H] \\
&u(x, 0, t) = u(x, H, t) = 0 \quad x \in [0, L] \\
&u(x, y, 0) = f(x, y) \\
&u_t(x, y, 0) = g(x, y).
\end{aligned}$$

is:

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \cos\left(c\sqrt{\lambda_{mn}} t\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) + \\ b_{mn} \sin\left(c\sqrt{\lambda_{mn}} t\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

where:

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2.$$

and

$$a_{mn} = \frac{4}{LH}\int_0^L \int_0^H f(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)\, dy\, dx.$$

and

$$b_{mn} = \frac{4}{LH}\int_0^L \int_0^H h(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)\, dy\, dx = \\ \frac{4}{LHc\sqrt{\lambda_{mn}}}\int_0^L \int_0^H g(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)\, dy\, dx.$$

□

Remark 7.5. A similar derivation allows us to construct the following proposition on the heat equation in the plane with homogeneous Dirichlet boundary conditions.

Proposition 7.6. *Assuming term-by-term differentiation is allowable, the solution to the heat equation with homogeneous Dirichlet boundary conditions in the plane:*

$$\begin{aligned}
&u_t = k\left(u_{xx} + u_{yy}\right)\\
&u(0, y, t) = u(L, y, t) = 0 \quad y \in [0, H]\\
&u(x, 0, t) = u(x, H, t) = 0 \quad x \in [0, L]\\
&u(x, y, 0) = f(x, y)
\end{aligned}$$

is:

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \exp\left[-k\lambda_{mn} t\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

where:

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2.$$

and

$$b_{mn} = \frac{4}{LH}\int_0^L \int_0^H f(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)\, dy\, dx.$$

$\square$

Example 7.7. Consider the two dimensional wave equation:

$$\begin{aligned}
&u_{tt} = \left(u_{xx} + u_{yy}\right)\\
&u(0, y, t) = u(\pi, y, t) = 0 \quad y \in [0, \pi]\\
&u(x, 0, t) = u(x, \pi, t) = 0 \quad x \in [0, \pi]\\
&u(x, y, 0) = \sin(2x)\sin(2y)\\
&u_t(x, y, 0) = 0.
\end{aligned}$$

Note that $c = 1$ and $L = H = \pi$. Applying Proposition 7.4 (and a little common sense) we see that:

$$a_{mn} = \begin{cases} 1 & \text{if } m = n = 2\\ 0 & \text{otherwise}\end{cases}$$

and $b_{mn} = 0$ for all m, n. We also note that:

$$\lambda_{2,2} = \left(\frac{2\pi}{\pi}\right)^2 + \left(\frac{2\pi}{\pi}\right)^2 = 8.$$

Therefore, the solution to the wave equation is:

$$u(x, y, t) = \sin(2x)\sin(2y)\cos(t\sqrt{8}).$$

The solution is illustrated in Fig. 7.1. It is easy to see that when $x = \pi/2$ or $y = \pi/2$, then $u(x, y, t)$ is constant (zero, in fact). These lines where $u(x, y, t)$ is constant are called nodal lines. In a physical context, they are positions where the membrane does not move at all.

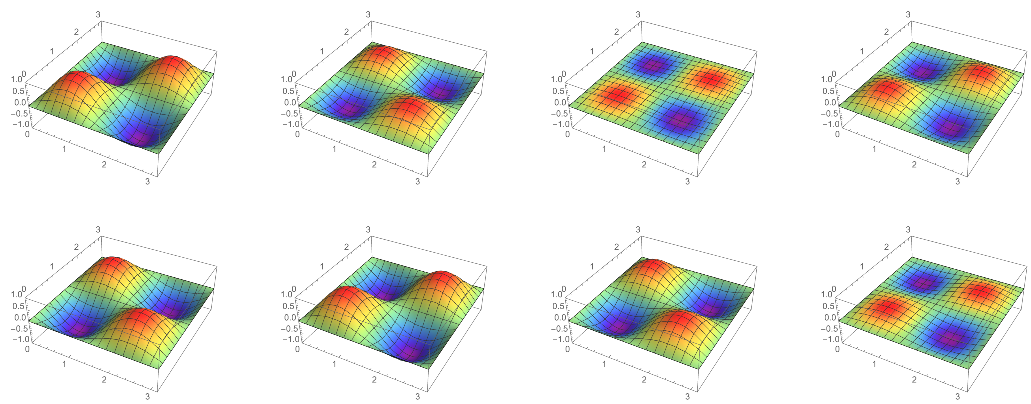

FIGURE 7.1. The solution of the 2D wave equation on the rectangle with Dirichlet boundary conditions. When $x = \pi/2$ or $y = \pi/2$, $u(x, y, t)$ is constant (i.e., the membrane does not move). These are called *nodal lines*.

2. The Poisson Equation on the Rectangle

Remark 7.8. Recall from Chapter 1 that the 2D Poisson equation is simply the forced or non-homogeneous Laplace equation:

$$\Delta u = f(x, y)$$

Using our knowledge of eigenfunction expansion (generalized to 2D) we can now solve this problem.

Proposition 7.9. *Consider the two-dimensional Poisson equation in the rectangle:*

$$\begin{aligned}
&\Delta u = f(x, y)\\
&u(x, 0) = g_b(x)\\
&u(x, H) = g_t(x)\\
&u(0, y) = g_l(y)\\
&u(L, y) = g_r(y).
\end{aligned}$$

Let $u_1(x, y)$ and $u_2(x, y)$ be solutions to the following PDE's:

$$\begin{aligned}
&\Delta u_1 = 0 && \Delta u_2 = f(x, y)\\
&u_1(x, 0) = g_b(x) && u_2(x, 0) = 0\\
&u_1(x, H) = g_t(x) && u_2(L, 0) = 0\\
&u_1(0, y) = g_l(y) && u_2(0, y) = 0\\
&u_1(L, y) = g_r(y). && u_2(H, y) = 0.
\end{aligned}$$

Then $u(x, y) = u_1(x, y) + u_2(x, y)$, $u_1(x, y)$ solves the given Laplace Equation and $u_2(x, y)$ solves the homogeneous Poisson equation with solution:

$$u_2(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right),$$

where:

$$c_{mn} = -\frac{4}{\lambda_{mn} LH} \int_0^L \int_0^L f(x,y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right)\, dy\, dx,$$

with

$$\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2.$$

PROOF. Assume that u_1 and u_2 solve the appropriate PDE's. Then:

$$\Delta(u_1 + u_2) = f(x,y)$$

as required. The resulting boundary conditions are also satisfied and thus we see that $u = u_1 + u_2$ is a solution to the Poisson problem. The exact form of u_1 can be obtained from Proposition 5.8. It now suffices to show that u_2 as given solves the Poisson problem with homogeneous Dirichlet boundary conditions.

Consider the function:

$$u_{mn}(x,y) = c_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

We will find c_{mn} so that:

$$u_2(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x,y)$$

solves the homogeneous Poisson problem. Differentiating we see:

$$\Delta u_{mn} = -c_{mn} \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2\right] \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) = -\lambda_{mn} u_{mn}.$$

But then:

$$\Delta\left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}\right] = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn} u_{mn} = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn} c_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

The right hand is a double Fourier decomposition and we are free to choose c_{mn} as we like. Therefore, we may choose c_{mn} so that:

$$-\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn} c_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) = f(x,y).$$

Let:

$$a_{mn} = -\lambda_{mn} c_{mn}.$$

Then:

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right)$$

Then:

$$a_{mn} = \frac{4}{LH} \int_0^L \int_0^H f(x,y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right)\, dy\, dx.$$

We conclude that:

$$c_{mn} = -\frac{4}{LH\lambda_{mn}} \int_0^L \int_0^H f(x,y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right)\, dy\, dx.$$

This completes the proof. □

Example 7.10. Consider the simple Poisson problem on the unit square $\Omega = [0,1] \times [0,1]$:

$$\begin{aligned}\Delta u &= \sin(\pi x)\sin(2\pi y)\\ u(\mathbf{x}) &= 0 \quad \forall \mathbf{x} \in \partial\Omega.\end{aligned}$$

It's clear that $c_{mn} = 0$ if $m \neq 1$ and $n \neq 2$. It suffices to find $c_{1,2}$. We have:

$$\lambda_{1,2} = \pi^2 + (2\pi)^2 = 5\pi^2.$$

Then:

$$c_{1,2} = -\frac{4}{5\pi^2}\int_0^1\int_0^1 \left[\sin(\pi x)\sin(2\pi y)\right]^2\,dy\,dx = -\frac{1}{5\pi^2}.$$

Then we deduce:

$$u(x,y) = -\frac{1}{5\pi^2}\sin(\pi x)\sin(2\pi y)$$

The resulting function is illustrated in Fig. 7.2.

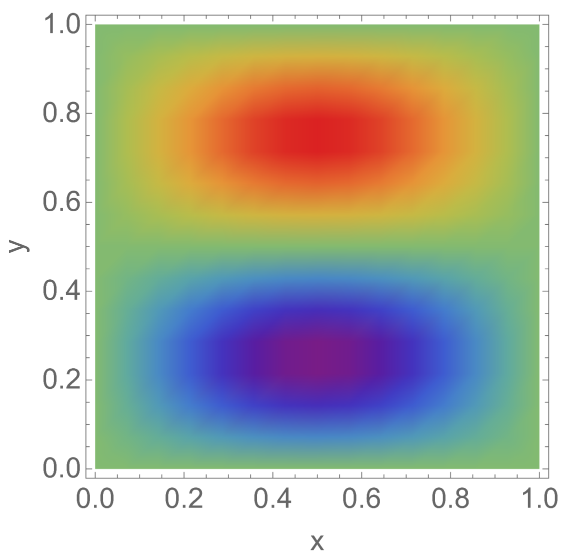

FIGURE 7.2. The solution to the Poisson equation when (e.g.) there is an external heat source $f(x,y) = \sin(\pi x)\sin(2\pi y)$ and equilibrium is achieved.

Derivation 7.11 (The Helmholtz Equation)**.** Consider the two dimensional wave equation again. This time, assume that the decomposition is given as:

$$u(x,y,t) = v(t)w(x,y).$$

Then we can write the wave equation as:

$$v''w = c^2 v\Delta w,$$

which separates into:

$$\frac{1}{c^2}\frac{v''}{v} = \frac{\Delta w}{w} = -\lambda$$

The PDE with boundary conditions:

$$\begin{aligned}
&\Delta w = -\lambda w \\
&w(0, y) = w(L, y) = 0 \quad y \in [0, H] \\
&w(x, 0) = w(x, H) = 0 \quad x \in [0, L]
\end{aligned}$$

is the homogeneous Helmholtz equation with homogeneous Dirichlet boundary conditions. From the work above, it is straightforward to see that solutions are eigenfunctions w_{mn} and eigenvalues λ_{mn} with:

$$w_{mn}(x, y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right),$$

and:

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2.$$

This is the two-dimensional analog of Lemma 2.15 and clearly results in a infinite family of eigenvalues and eigenfunctions, just as in our work in Sturm-Liouville theory, which we will see generalizes to higher dimensions, as we see in the next section.

Remark 7.12. The Helmholtz equation is a generalization of the eigenvalue problems we've already seen in Lemmas 2.15, 2.18 and 2.21.

3. Generalized Sturm-Liouville Problems

Remark 7.13. We can generalize the previous result just as we did in the case of Sturm-Liouville problems.

THEOREM 7.14. *Let $p, q, \sigma : \mathbb{R}^n \to \mathbb{R}$ be known functions and let $u(\mathbf{x})$ be an unknown function. Let $a, b \in \mathbb{R}$. Consider the generalized eigenvalue PDE with Robin boundary conditions defined on the region Ω:*

$$\begin{aligned}
&\nabla \cdot (p(\mathbf{x})\nabla u) + q(\mathbf{x})u + \lambda\sigma(x)u = 0 \\
&au(\mathbf{x}) + b\nabla u(\mathbf{x}) \cdot \mathbf{n} = 0 \quad \forall \mathbf{x} \in \partial\Omega.
\end{aligned}$$

Then the following statements hold:

(1) *There are an infinite number of eigenvalues λ all of which are real. Each eigenvalue λ corresponds to an eigenfunction $u_\lambda(\mathbf{x})$.*
(2) *There is a smallest eigenvalue λ_{min} but no largest eigenvalue.*
(3) *The eigenfunctions are complete in the sense that if $f(\mathbf{x})$ is piecewise C^2 then we have the generalized Fourier series presentation:*

$$f(\mathbf{x}) \sim \sum_\lambda a_\lambda u_\lambda(\mathbf{x}).$$

(4) *If $\lambda_1 \neq \lambda_2$, then:*

$$\int_\Omega u_{\lambda_1}(\mathbf{x})u_{\lambda_2}(\mathbf{x})\sigma(\mathbf{x})\,d\Omega = 0.$$

That is, the eigenfunctions of distinct eigenvalues are orthogonal with respect to the inner product:

$$\langle f, g\rangle_\sigma = \int_\Omega f(\mathbf{x})g(\mathbf{x})\sigma(\mathbf{x})\,d\Omega$$

(5) *If* $\lambda_1 = \lambda_2$ *having (at least) two distinct eigenfunctions* u_{λ_1} *and* u_{λ_2}*, then it is possible to choose* u_{λ_2} *so that these eigenfunctions are still orthogonal.*
(6) *Finally, an eigenvalue and corresponding eigenfunction are related by the Raleigh quotient:*

$$\lambda = \frac{-\int_{\partial\Omega} u_\lambda \nabla u_\lambda \cdot \mathbf{n}\, dS + \int_\Omega |\Delta u|^2\, d\Omega}{\int_\Omega u_\lambda^2\, d\Omega},$$

where we assume $S = \partial\Omega$*.*

□

Remark 7.15. The proof of the simpler parts of the theorem are (essentially) identical to the proofs already seen in Chapter 6, but using the Green's Identities derived in Chapter 5 to prove that the Laplace equation is well-posed. We can however deduce a few results on the Laplacian and generalized Fourier series in higher dimensions.

Corollary 7.16. *Consider the Helmholtz equation with Dirichlet boundary conditions:*

$$\Delta u = -\lambda u$$
$$u(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \partial\Omega.$$

Then the operator Δ *is self-adjoint under the resulting inner-product.*

Proof. In our general framework, we have $p(\mathbf{x}) = \sigma(\mathbf{x}) = 1$ and $q(\mathbf{x}) = 0$. Consider:

$$\langle u, \Delta v\rangle - \langle \Delta u, v\rangle = \int_\Omega u\Delta v\, d\Omega - \int_\Omega v\Delta u\, d\Omega.$$

Apply Green's Second Identity (Theorem 5.49) to see that:

$$\langle u, \Delta v\rangle - \langle \Delta u, v\rangle = \int_S [u\, \mathrm{grad}(v) - v\, \mathrm{grad}(u)] \cdot \mathbf{n}\, dS.$$

Since $u(\mathbf{x}) = v(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\Omega$, the right-hand-side must be 0. Therefore we have shown that the self-adjoint criterion:

$$\langle u, \Delta v\rangle = \langle \Delta u, v\rangle$$

is satisfied. □

Corollary 7.17. *Consider the generalized eigenvalue PDE with Robin boundary conditions defined on the region* Ω*:*

$$\Delta u + \lambda u = 0$$
$$au(\mathbf{x}) + b\nabla u(\mathbf{x}) \cdot \mathbf{n} = 0 \quad \forall \mathbf{x} \in \partial\Omega.$$

Let $f(\mathbf{x})$ *be a piecewise* C^2 *function with generalized Fourier decomposition:*

$$f(\mathbf{x}) \sim \sum_\lambda a_\lambda u_\lambda(\mathbf{x}).$$

Then:

$$a_\lambda = \frac{\int_\Omega f(\mathbf{x}) u_\lambda(\mathbf{x})\, d\Omega}{\int_\Omega [u_\lambda(\mathbf{x})]^2\, d\Omega}. \tag{7.1}$$

EXERCISE 44. Prove Corollary 7.17. [Hint: Start with the generalized Fourier decomposition. Multiply both sides by $u_\lambda(\mathbf{x})$ and integrate. You can also think of this as taking an inner product with $u_\lambda(\mathbf{x})$ on both sides.]

4. Wave Equation on a Circular Membrane

Remark 7.18. We now turn our attention to the Wave Equation in a circular membrane with Dirichlet boundary conditions, which is essentially the drumhead problem; i.e., model the behavior of a vibrating drumhead as a result of a disturbance given by the initial conditions. Since we assume no friction and no gravity this is not a true model. The drumhead will vibrate forever, but it is an excellent illustration of the use of Bessel functions as we will see.

Derivation 7.19. Let $u(r, \theta, t)$ be the position of the drumhead at time t and polar coordinate (r, θ). Assume the drum has radius R.

$$\begin{aligned}
&u_{tt} = c^2 \Delta u = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) \\
&u(r, -\pi) = u(r, \pi) \\
&u_\theta(r, -\pi) = u_\theta(r, \pi) \\
&u(R, \theta) = 0 \quad \forall \theta \in [-\pi, \pi] \\
&u(r, \theta, 0) = f(r, \theta) \\
&u_t(r, \theta, 0) = g(r, \theta)
\end{aligned}$$

Assume that we can write $u(r, \theta, t) = v(t) w(r) p(\theta)$. Then we have:

$$\frac{1}{c^2} \frac{v''(t)}{v(t)} = \frac{w''(r)}{w(r)} + \frac{1}{r} \frac{w'(r)}{w(r)} + \frac{1}{r^2} \frac{p''(\theta)}{p(\theta)} = -\lambda.$$

As before, the time ODE is given by:

$$v'' + c^2 \lambda v = 0.$$

The space PDE can be written as:

$$r^2 \frac{w''(r)}{w(r)} + r \frac{w'(r)}{w(r)} + r^2 \lambda = -\frac{p''(\theta)}{p(\theta)} = \mu$$

This gives an angle ODE and a radial ODE:

$$\begin{aligned}
&p'' + \mu p = 0 \\
&p(-\pi) = p(\pi),
\end{aligned}$$

where the boundary conditions arise from the periodic boundary conditions on the PDE. The radial ODE is:

$$\begin{aligned}
&r^2 w'' + r w' + \left(r^2 \lambda - \mu \right) w = 0 \\
&w(R) = 0 \\
&w(0) \text{ bounded},
\end{aligned}$$

where the boundary conditions arise from the boundary condition on the PDE and common sense.

Angle ODE:
Solving the angular ODE using Lemma 2.21 we see a family of solutions arise with:

$$p_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$$

and $\mu_n = n^2$. (This is really $(n\pi/L)^2$ but $L = \pi$.) There is also an $n = 0$ constant solution $p_0 = a_0$, which we will absorb into the $\cos(nx)$ term for readability.
Radial ODE:
Using this solution in the radial ODE yields

$$\begin{aligned}
&r^2 \frac{d^2 w}{dr^2} + r\frac{dw}{dr} + \left(r^2\lambda - n^2\right) w = 0 \\
&w(R) = 0 \\
&w(0) \text{ bounded},
\end{aligned}$$

Let $z = \sqrt{\lambda} r$. Then:

$$\frac{dw}{dr} = \frac{dw}{dz}\frac{dz}{dr} = \sqrt{\lambda}\frac{dw}{dz}$$

and likewise

$$\frac{d^2 w}{dr^2} = \lambda \frac{d^2 w}{dz^2}.$$

Using the fact that $z^2 = \lambda r^2$ we can write the radial ODE as:

$$\frac{z^2}{\lambda}\left(\lambda \frac{d^2 w}{dz^2}\right) + \frac{z}{\sqrt{\lambda}}\left(\sqrt{\lambda}\frac{dw}{dz}\right) + \left[\left(\frac{z^2}{\lambda}\right)\lambda - n^2\right] w = 0 \implies$$
$$z^2 w'' + z w' + (z^2 - n^2) w = 0.$$

This is Bessel's equation. We know from Example 6.36 that the general solution is:

$$w(z) = c_1 J_n(z) + c_2 Y_n(z),$$

or using our defined value for z:

$$w(r) = c_1 J_n(\sqrt{\lambda} r) + c_2 Y_n(\sqrt{\lambda} r).$$

The fact that we require $w(r)$ to be bounded at $r = 0$ immediately implies that $c_2 = 0$. The other boundary condition $w(R) = 0$ implies that we choose λ so that:

$$J_n(\sqrt{\lambda} R) = 0.$$

That is, we must solve for λ numerically. Recall from Example 6.36 that the Bessel function of the first kind $J_m(x)$ has a countably infinite number of zeros. Therefore we can number them and let λ_{mn} be the m^{th} zero of $J_n(\sqrt{\lambda} R)$.
Time ODE:
We can now solve the original time ODE as we did in the case of the rectangular membrane to see that:

$$v_{mn}(t) = a_{mn} \cos(c\sqrt{\lambda_{mn}} t) + b_{mn} \sin(c\sqrt{\lambda_{mn}} t),$$

where λ is the value just computed numerically. Combining all terms together yields:

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{mn}}r)\left[a_{mn}\cos(n\theta) + b_{mn}\sin(n\theta)\right]\cos(c\sqrt{\lambda_{mn}}t) + \\ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{mn}}r)\left[\alpha_{mn}\cos(n\theta) + \beta_{mn}\sin(n\theta)\right]\sin(c\sqrt{\lambda_{mn}}t)$$

The coefficients can be derived as in other cases using a generalized Fourier decomposition. For example:

$$u(r,\theta,0) = f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{mn}}r)\left[a_{mn}\cos(n\theta) + b_{mn}\sin(n\theta)\right].$$

Because we are working on a circle, we can define the inner product:

$$\langle f, h\rangle = \int_0^R \int_{-\pi}^{\pi} f(r,\theta)h(r,\theta)\,d\theta\,dr.$$

For simplicity, assume $g(r,\theta) = 0$. By our generalized Sturm-Liouville results we know that:

$$a_{mn} = \frac{\left\langle f, J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta)\right\rangle}{\left\langle J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta), J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta)\right\rangle}$$
$$b_{mn} = \frac{\left\langle f, J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta)\right\rangle}{\left\langle J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta), J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta)\right\rangle}.$$

and $\alpha_{mn} = \beta_{mn} = 0$. When $g(r,\theta) \neq 0$, the values of these coefficients can be derived in a similar way. Note:

$$u_t(r,\theta,0) = g(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{mn}}J_n(\sqrt{\lambda_{mn}}r)\left[\alpha_{mn}\cos(n\theta) + \beta_{mn}\sin(n\theta)\right]$$

We conclude:

$$\alpha_{mn} = \frac{1}{c\sqrt{\lambda_{mn}}}\frac{\left\langle g, J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta)\right\rangle}{\left\langle J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta), J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta)\right\rangle}$$
$$\beta_{mn} = \frac{1}{c\sqrt{\lambda_{mn}}}\frac{\left\langle g, J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta)\right\rangle}{\left\langle J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta), J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta)\right\rangle}.$$

Remark 7.20. Using properties of the Bessel function, it is possible to simplify the expressions for a_{mn}, b_{mn}, α_{mn} and β_{mn}. The interested reader should consult [**Hab03**] or [**Asm16**].

Example 7.21. There is a point where computing solutions by hand becomes infeasible (even with Mathematica) and this is the point. We use Mathematica's differential equation solver to solve the following wave equation:

$$
\begin{aligned}
&u_{tt} = \Delta u\\
&u(r,\theta,0) = (1-r^2)r\sin(\theta)\\
&u_t(r,\theta,0) = 0\\
&u(r,-\pi) = u(r,\pi)\\
&u_\theta(r,-\pi) = u_\theta(r,\pi)\\
&u(1,\theta) = 0 \quad \forall\theta\in[-\pi,\pi]
\end{aligned}
$$

In this problem we have $R = 1$. The simplest command to solve this problem is:

```
NDSolve[{D[u[x, y, t], {t, 2}] == Laplacian[u[x, y, t], {x, y}],
   u[x, y, 0] == (1 - x^2 - y^2)*Sqrt[x^2 + y^2]*
     Sin[ArcTan[x, y]], ((D[u[x, y, t], t]) /. t -> 0) == 0,
   DirichletCondition[u[x, y, t] == 0, True]},
  u, {x, y} \[Element] Disk[{0, 0}, 1], {t, 0, 10}];
```

Notice we are actually using Cartesian coordinates but telling the system to solve in the unit disk. The resulting solution is illustrated in Fig. 7.3.

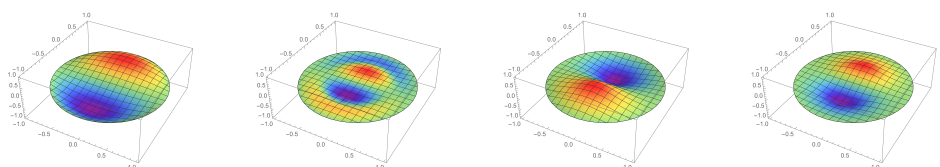

FIGURE 7.3. The dynamics of a circular membrane under the wave equation.

5. 3D Waves & Huygen's Principle

Remark 7.22. In this final section, we will discuss three dimensional spherical waves in free space (i.e., $\mathbb{R}^3$). We will show (but not prove) Kirchoff's Formula, which was really derived by Poisson. This formula is the three-dimensional analog of D'Almbert's Formula, but can be used to derive a solution to the wave equation in two-dimensional free space.

THEOREM 7.23. *Consider the three dimensional wave equation in free space:*

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2\Delta u\\
&u(\mathbf{x},0) = f(\mathbf{x})\\
&u_t(\mathbf{x},0) = g(\mathbf{x}).
\end{aligned}
$$

Then $u(\mathbf{x},t)$ is given by:

$$
u(\mathbf{x}_0,t) = \frac{1}{4\pi c^2 t}\int_S g(\mathbf{x})\,dS + \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_S f(\mathbf{x})\,dS\right], \tag{7.2}
$$

where S is a two-dimensional surface:

$$
S = \{\mathbf{x}\in\mathbb{R}^3 : \|\mathbf{x}-\mathbf{x}_0\| = ct\}.
$$

That is, S is the surface of a sphere with radius ct centered at $\mathbf{x}_0$.

Remark 7.24. The integrals in Eq. (7.2) are surface integrals. The proof of this result uses a method called the *method of spherical means* and is not terrily difficult, but is a bit long to include. The proof can be found in [**Str08**]. An alternate analysis can be found in [**Olv14**].

Derivation 7.25 (Solution in Two Dimensions)**.** Consider the wave equation in two-dimensional free space and for simplicity assume that $g(\mathbf{x}) = 0$. Also, for simplicity, let $\mathbf{x}_0 = (0, 0, 0)$. To correct this later, just subtract x_0 from x and y_0 from y in the final formulas. (That is translate your solution.)

We can think of the initial condition $f(\mathbf{x})$ as just being a function of x, y and z as in Theorem 7.23 with no dependence on z. The resulting solution is a function $u(\mathbf{x}, t)$ that can be derived from Eq. (7.2) but has no dependence on z.

Using Eq. (7.2) we have:

$$u(0,0,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{x^2+y^2+z^2=c^2t^2} f(x,y)\,dS\right].$$

Since $f(x, y)$ does not depend on z, we can consider the upper-hemisphere H of the surface S and by a symmetry argument we have:

$$\frac{1}{4\pi c^2 t}\int_{x^2+y^2+z^2=c^2t^2} f(x,y)\,dS = \frac{2}{4\pi c^2 t}\int_H f(x,y)\,dH. \tag{7.3}$$

The upper hemisphere of the surface can be written as:

$$z = H(x,y) = \sqrt{c^2t^2 - x^2 - y^2}.$$

Define the region in the plane:

$$R = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq c^2t^2\right\}.$$

This is the projection of H onto $\mathbb{R}^2$. We know from Vector Calculus that we can re-write the surface integral in Eq. (7.3) as the (simple) double integral:

$$\frac{2}{4\pi c^2 t}\int_H f(x,y)\,dH = \frac{1}{2\pi c^2 t}\int_R f(x,y)\sqrt{1 + \left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2}\,dx\,dy$$

Differentiating H we see:

$$\begin{aligned}(H_x)^2 &= \frac{x^2}{c^2t^2 - x^2 - y^2}\\ (H_y)^2 &= \frac{y^2}{c^2t^2 - x^2 - y^2}.\end{aligned}$$

Then:

$$1 + (H_x)^2 + (H_y)^2 = \frac{c^2t^2}{c^2t^2 - x^2 - y^2}.$$

From this we see:

$$\begin{aligned}u(0,0,t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c^2 t}\int_R f(x,y)\frac{ct}{\sqrt{c^2t^2 - x^2 - y^2}}\,dx\,dy\right] &=\\ \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_R \frac{f(x,y)}{\sqrt{c^2t^2 - x^2 - y^2}}\,dx\,dy\right]&.\end{aligned}$$

The analysis is identical for the term with $g(\mathbf{x})$ when $g(\mathbf{x}) \neq 0$. We obtain the complete two dimensional solution:

$$u(0,0,t) = \frac{1}{2\pi c}\int_R \frac{g(x,y)}{\sqrt{c^2t^2 - x^2 - y^2}}\,dx\,dy + \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_R \frac{f(x,y)}{\sqrt{c^2t^2 - x^2 - y^2}}\,dx\,dy\right].$$

In general for a point (x_0, y_0) we have:

$$u(x_0,y_0,t) = \frac{1}{2\pi c}\int_R \frac{g(x,y)}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}\,dx\,dy + \\ \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_R \frac{f(x,y)}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}\,dx\,dy\right],$$

where $R = \{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq c^2t^2\}$.

Remark 7.26 (Huygen's Principle)**.** Let us now compare the solution in three dimensions and the solution in two dimensions. Notice that the value of $u(x,y,z,t)$ is entirely a function of the *surface* integral on the sphere S. This means that the initial values $f(\mathbf{x})$ and $g(\mathbf{x})$ influence the solution *only* on that surface. That is, information from the initial conditions at point $(x,y,z,t) = (0,0,0,0)$ is available at point (x,y,z) only when $x^2+y^2+z^2 = c^2t^2$. Thus, all information is carried on the wavefront spreading out at the speed of c. Because of this:

- In free space, we hear sound without echos.
- We see sharp images.
- We have a sense of sound/light causality (i.e., we don't see the past).

This idea that information is carried on the wave front in three dimensions is called *Huygen's principle* and it is valid in all odd dimensions above 1.

Contrast this with the two-dimensional solution. Here the integral is over the region R. This means that $u(x,y,t)$ is determined by information from *inside* the region $x^2+y^2 \leq c^2t^2$. That is, in two dimensions, information is contained inside the wave itself and so there is no sharp transmission of information. You see this in a pond, where ripples continue to spread even after an initial disturbance and only die down asymptotically. Thus, in Flatland, the Math 412 lecture begins…and keeps on going forever.

Remark 7.27 (Closing Remarks)**.** Three dimensional PDE's can be solved using separation of variables in much the same way as two-dimensional problems. There are just more ODE's to solve and the Laplacian becomes more complicated (especially in spherical coordinates). The most interesting results come when studying problems on the sphere in which case spherical harmonics and Legendre polynomials (as opposed to Bessel functions) emerge. (This is why Legendre polynomials come up in solutions to Schrödinger's equation with a central force.) The interested student can and should consult Haberman [**Hab03**] or Olver [**Olv14**] or [**Asm16**] Asmar (or any other of a number of books) for details.

CHAPTER 8

Fourier Transform Methods & Fundamental Solutions

1. The Fourier Transform

Remark 8.1. Consider a function $f(x)$ that is defined on $\mathbb{R}$, rather than just an interval $[-L, L]$. Such a function may not have a Fourier series representation because it is defined on the whole real line, but we can generalize the idea of a Fourier decomposition by replacing our sum with an integral.

Definition 8.2 (Absolutely Integrable)**.** A function $f(x)$ is absolutely integrable on a an interval $I = (a, b)$ if:

$$\int_a^b |f(x)|\,dx < \infty.$$

THEOREM 8.3 (Fourier Integral Representation). *Let $f(x)$ be a piecewise C^2 on every finite interval and absolutely integrable on $\mathbb{R}$. Then there are functions $A(\omega)$ and $B(\omega)$ such that:*

$$f(x) \sim \int_0^\infty A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\,d\omega,$$

where for all $\omega \geq 0$ we have:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx.$$

Here $\sim$ denotes equality when $f(x)$ is continuous or the that the integral converges to:

$$\frac{f(x^+) + f(x^-)}{2},$$

if x is a point of discontinuity.

Example 8.4. This example comes from [**Asm16**]. Consider the function:

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We can compute:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx = \int_{-1}^{1}\cos(\omega x)\,dx = \frac{1}{\pi\omega}\sin(\omega x)\Big|_{-1}^{1} = \frac{2}{\pi}\frac{\sin(\omega)}{\omega}.$$

We can also compute $B(\omega) = 0$. The result is we can write:

$$f(x) \sim \int_0^\infty \frac{2}{\pi}\frac{\sin(\omega)}{\omega}\cos(\omega x)\,d\omega.$$

From the Fourier integral theorem, we can see that:

$$\int_0^\infty \frac{2}{\pi}\frac{\sin(\omega)}{\omega}\cos(\omega x)\,d\omega = \begin{cases} 1 & \text{if } -1 < x < 1 \\ \frac{1}{2} & \text{if } x = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

We note that the integral:

$$\int_0^\infty \frac{\sin(\omega)}{\omega},$$

cannot be evaluated using elementary functions. However, using the information from the Fourier integral representation we can set $x = 0$ and see that:

$$f(0) = 1 = \int_0^\infty \frac{2}{\pi}\frac{\sin(\omega)}{\omega}\,d\omega.$$

Thus we can evaluate the integral:

$$\int_0^\infty \frac{\sin(\omega)}{\omega}\,d\omega = \frac{\pi}{2}.$$

This is called the Dirichlet integral.

Derivation 8.5 (Fourier Transform)**.** Substitute the expressions for $A(\omega)$ and $B(\omega)$ into the Fourier integral theorem:

$$f(x) \sim \int_0^\infty \left[\frac{1}{\pi}\int_{-\infty}^\infty f(y)\cos(\omega y)\,dy\right]\cos(\omega x) + \left[\frac{1}{\pi}\int_{-\infty}^\infty f(y)\sin(\omega y)\,dx\right]\sin(\omega x)\,d\omega,$$

where we use a different dummy variable y for the integrals defining $A(\omega)$ and $B(\omega)$. We can now write this as:

$$f(x) \sim \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\left[\cos(\omega y)\cos(\omega x) + \sin(\omega y)\sin(\omega x)\right]\,dy\,d\omega.$$

Now apply the trigonometric identity from Eq. (5.19)

$$f(x) \sim \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos\left[\omega(x-y)\right]\,dy\,d\omega.$$

Now use the definition of the cosine function Corollary 5.26 to see:

$$f(x) \sim \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \frac{1}{2}f(y)\left(e^{i\omega(x-y)} + e^{-i\omega(x-y)}\right)\,dy\,d\omega.$$

Break the integral into two parts:

$$f(x) \sim \frac{1}{2\pi}\left[\int_0^\infty \int_{-\infty}^\infty f(y)e^{i\omega(x-y)}\,dy\,d\omega + \int_0^\infty \int_{-\infty}^\infty f(y)e^{-i\omega(x-y)}\,dy\,d\omega\right].$$

Rewrite the second integral with the substitution $\upsilon = -\omega$ to see:

$$\int_0^\infty \int_{-\infty}^\infty f(y)e^{-i\omega(x-y)}\,dy\,d\omega = -\int_0^{-\infty}\int_{-\infty}^\infty f(y)e^{-i\upsilon(x-y)}\,dy\,d\upsilon =$$
$$\int_{-\infty}^0 \int_{-\infty}^\infty f(y)e^{i\upsilon(x-y)}\,dy\,d\upsilon.$$

Since υ is a dummy variable, we can change it back to ω and see:

$$f(x) \sim \frac{1}{2\pi}\left[\int_0^\infty \int_{-\infty}^\infty f(y)e^{i\omega(x-y)}\,dy\,d\omega + \int_{-\infty}^0 \int_{-\infty}^\infty f(y)e^{i\omega(x-y)}\,dy\,d\omega\right] =$$
$$\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(y)e^{i\omega(x-y)}\,dy\,d\omega.$$

Now we pull the integral apart:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega x}\left[\int_{-\infty}^\infty e^{-i\omega y}f(y)\,dy\right]d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}\left[\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-i\omega y}f(y)\,dy}_{\hat{f}(\omega)}\right]d\omega.$$

We now have the following definition.

Definition 8.6 (Fourier/Inverse Fourier Transform)**.** The integral transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-i\omega x}f(x)\,dx$$

is the Fourier transform (assuming the integral exists) and the inverse Fourier transform is:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}\hat{f}(\omega)\,d\omega$$

is the inverse Fourier transform.

Remark 8.7. The multipliers $1/\sqrt{2\pi}$ are a little arbitrary as you can tell from the derivation. In fact the definition of the Fourier transform may vary a little (in terms of coefficients) from book to book. This means you need to be a little careful when you use a table to determine which version of the Fourier transform was used in making the table. That said, there are many such tables on the Internet and you can now compute Fourier transforms with computer algebra systems.

Remark 8.8. For the remainder of this chapter, we assume that $f(x) \to 0$ and $|x| \to \infty$ and that $f(x)$ is piecewise C^2 on every finite interval and absolutely integrable. This will take care of the problems of the Fourier transform existing. The variable ω is like a frequency and generalizes the idea of a Fourier decomposition.

Remark 8.9. There are several ways to denote a Fourier transform. Assuming $f(x)$ is a function we can denote the Fourier transform as $\hat{f}(\omega)$. Another way to denote this is: $\mathcal{F}[f(x)]$. The second method is convenient because we can denote the inverse transform as $\mathcal{F}^{-1}(\hat{f}(\omega)) = f(x)$.

2. Properties of the Fourier Transform

Proposition 8.10. *The Fourier transform is a linear. That is, given two function $f(x)$ and $g(x)$ and a scalar α we have:*

$$\mathcal{F}[\alpha f(x) + g(x)] = \alpha \hat{f}(\omega) + \hat{g}(\omega).$$

□

EXERCISE 45. Prove that the Fourier transform is linear.

THEOREM 8.11. *Suppose that $f(x)$ is C^2 and both $f(x)$ and $f'(x)$ are absolutely integrable and as $|x| \to \infty$, $f(x) \to 0$. Then:*

$$\mathcal{F}[f'(x)] = i\omega \hat{f}(\omega). \tag{8.1}$$

PROOF. Consider the Fourier transform of $f'(x)$:

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f'(x)\, dx.$$

Proceed by integration by parts: $u = e^{-i\omega x}$, $dv = f'(x)dx$, $v = f(x)$, $du = -i\omega e^{-i\omega} dx$. We conclude:

$$\mathcal{F}[f'(x)] = e^{-i\omega x} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\omega e^{-i\omega} f(x)\, dx$$

The first term goes to zero since we assumed that as $|x| \to \infty$, $f(x) \to 0$. Then we can write:

$$\mathcal{F}[f'(x)] = i\omega \int_{-\infty}^{\infty} e^{-i\omega} f(x)\, dx = i\omega \hat{f}(\omega).$$

ha □

Corollary 8.12. *Suppose $f(x)$ is C^n and all derivatives are absolutely integrable and $f^{(k)} \to 0$ as $|x| \to \infty$ for all $0 \le k \le n$. Then:*

$$\mathcal{F}[f^{(n)}(x)] = (i\omega)^n \mathcal{F}[f(x)].$$

□

EXERCISE 46. Prove Corollary 8.12. [Hint: Use induction.]

THEOREM 8.13. *Suppose $f(x)$ and $xf(x)$ are absolutely integrable, then*

$$\mathcal{F}[xf(x)] = i\hat{f}'(\omega).$$

PROOF. From the Fourier transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx,$$

we can compute the derivative:

$$\hat{f}'(\omega) = \frac{d}{d\omega}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ixe^{-i\omega x} f(x)\, dx =$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ie^{-i\omega x}(xf(x))\, dx = -i\mathcal{F}[xf(x)].$$

Multiply both sides by i to see:

$$i\hat{f}'(\omega) = \mathcal{F}[xf(x)].$$

□

Corollary 8.14. *If $f(x)$ and $x^n f(x)$ are absolutely integrable, then:*

$$\mathcal{F}[x^n f(x)] = i^n \hat{f}^{(n)}(\omega).$$

□

EXERCISE 47. Prove Corollary 8.14. [Hint: Use induction.]

3. Convolutions

Definition 8.15. Consider two functions $f(x)$ and $g(x)$, then the *convolution* of the two functions:

$$(f * g)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z-s)g(s)\,ds$$

Remark 8.16. In some texts, the coefficient $1/\sqrt{2\pi}$ is omitted. The coefficients of the Fourier transform also vary by text.

Remark 8.17. The convolution is a strange binary operation on functions. Effectively, it translates the function $f(s)$ z units to the right to obtain $f(s-z)$ and then flips the function about the $x = z$ to obtain $h(s) = f(-(s-z)) = f(z-s)$. We then take the inner product of $h(s)$ with $g(s)$. When $h(s)$ and $g(s)$ agree, the inner product is large.

Some texts try to explain this as an averaging process but this never seems as clear as the authors intend. An intuitive way to think about it is that the shape of $f(s)$ modifies the shape of $g(s)$ in a very particular way.

Example 8.18. Consider the function:

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We will compute $f * f$. It is first helpful to compute $f(z-s)f(s)$. We can divide this up into cases using Fig. 8.1. Using the figure, we see that:

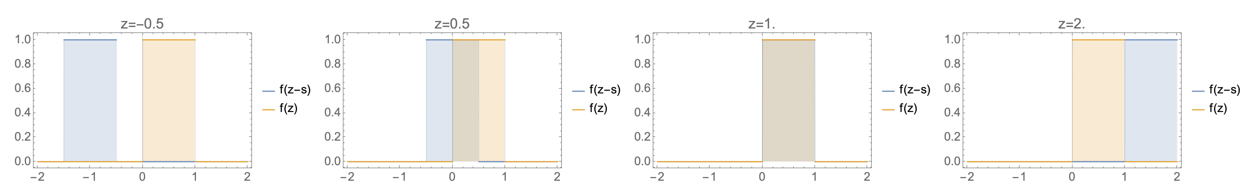

FIGURE 8.1. The two functions involved in convolving $f(x)$ with itself.

(1) If $x < 0$, then $f(x-s)f(x) = 0$.
(2) If $x > 2$, then $f(x-s)f(x) = 0$.
(3) If $x \in [0, 1]$ then:

$$f(x-s)f(x) = \begin{cases} 1 & \text{if } 0 \leq s \leq x \\ 0 & \text{otherwise} \end{cases}$$

(4) The tricky part is when $x \in [1,2]$. Using the figure (and imagining the blue plot sliding rightward), we see that:

$$f(x-s)f(x) = \begin{cases} 1 & \text{if } x-1 \leq s \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now we can compute

$$(f * f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z-s)f(s)\, ds.$$

As before:

(1) If $x < 0$, then $(f * f)(x) = 0$.
(2) If $x > 2$, then $(f * f)(x) = 0$.
(3) If $x \in [0,1]$ then:

$$(f * f)(x) = \frac{1}{\sqrt{2\pi}} x$$

(4) Now when $x \in [1,2]$, it's helpful to compute the integral:

$$\int_{-\infty}^{q} \begin{cases} 1 & \text{if } x-1 \leq s \leq 1 \\ 0 & \text{otherwise} \end{cases} ds = \begin{cases} 0 & \text{if } q < x-1 \\ q-(x-1) & \text{if } x-1 \leq q \leq 1 \\ 2-x & \text{otherwise.} \end{cases}$$

The middle result arises because when $q = x-1$, the integral would be exactly zero since we're just integrating right up to the point of intersection. Now we can let $q \to \infty$ to see that:

$$(f * f)(x) = 2 - x.$$

Thus the convolution is:

$$(f * f)(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

This is illustrated in Fig. 8.2.

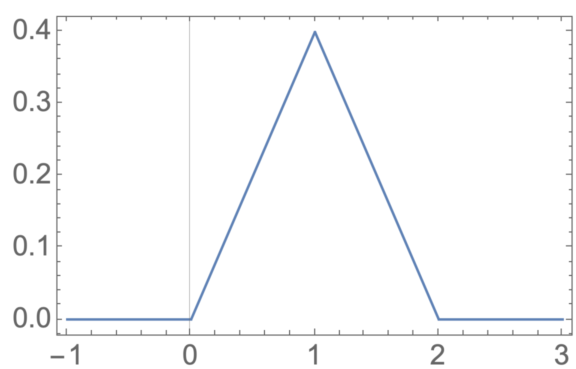

FIGURE 8.2. The convolution of f with itself creates a triangle function.

Proposition 8.19. *Convolution is commutative. That is* $(f * g)(x) = (g * f)(x)$.

Proof. Compute:

$$(f * g)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z-s)g(s)\,ds.$$

Let $u = z - s$ and $du = -ds$ Then $s = z - u$. We have:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z-s)g(s)\,ds = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} -f(u)g(z-u)\,du =$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(z-u)f(u)\,du = (g * f)(z).$$

□

Theorem 8.20. *Let f and g be two functions with Fourier transforms. Then:*

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

That is, Fourier transforms turn convolutions into multiplications.

Proof. Compute the Fourier transform of $f * g$:

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x}(f * g)(x)\,dx =$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-s)g(s)\,ds\right] dx$$

Bring the term $e^{-i\omega x}$ under the second integral (it is not a function of s), so it is constant. Notice: $e^{-i\omega x} = e^{-i\omega(x-s)} \exp^{-i\omega s}$. We also interchange the order of integration and rewrite this as:

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x-s)g(s)\,ds\right] dx =$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(x-s)} e^{-i\omega s} f(x-s)g(s)\,dx\right] ds$$

Let $u = x - s$ and $du = dx$. Then we have:

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega u} e^{-i\omega s} f(u)g(s)\,du\right] ds =$$
$$\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega u} f(u)\,du\right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega s} g(s)\,ds\right] = \hat{f}(\omega)\hat{g}(\omega).$$

□

4. The Gaussian Function

Lemma 8.21. *The function:*

$$u(x) = A \exp\left(-\frac{ax^2}{2}\right) \tag{8.2}$$

solves the differential equation:

$$u' + axu = 0.$$

for some constant of integration A. □

EXERCISE 48. Prove Lemma 8.21.

Remark 8.22. For the rest of this section, we refer to the function Eq. (8.2) as a Gaussian function. We now use this result to derive the Fourier transform of this function.

Remark 8.23. We will use the following fact from two-dimensional calculus. Let $f(x, y)$ be a function in Cartesian coordinates and let $f(r, \theta)$ be the same function expressed in polar coordinates. Then:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_{0}^{2\pi}\int_{0}^{\infty} f(r, \theta) r\, dr\, d\theta$$

If you haven't seen the proof, there's actually a really nice one on Wikipedia under the Jacobian entry.

Lemma 8.24. *Suppose $a > 0$, then:*

$$\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx = \frac{\sqrt{2\pi}}{\sqrt{a}}.$$

PROOF. Consider the integral product:

$$\left[\int_{-\infty}^{\infty} \exp\left(-x^2\right)\, dx\right]\left[\int_{-\infty}^{\infty} \exp\left(-y^2\right)\, dy\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-(x^2 + y^2)\right]\, dx\, dy$$

In polar coordinates, we can rewite the integral on the right as:

$$\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

Let $u = r^2$ and $du = 2rdr$. Then we have:

$$\int e^{-r^2} r\, dr = \int \frac{1}{2} e^{-u}\, du = -\frac{1}{2} e^{-u}.$$

Substituting into the integral yields:

$$\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\, dr\, d\theta = \int_{0}^{2\pi} \left[-\frac{1}{2} e^{-r^2}\right]\Bigg|_{0}^{\infty} d\theta = \int_{0}^{2\pi} \frac{d\theta}{2} = \pi.$$

Then it follows:

$$\left[\int_{-\infty}^{\infty} \exp\left(-x^2\right)\, dx\right]^2 = \pi$$

Therefore:

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right)\, dx = \sqrt{\pi}$$

To complete the proof, consider the integral:

$$\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx.$$

Let:

$$u = \sqrt{\frac{a}{2}} x.$$

Then using this substitution, the integral can be rewritten as:

$$\int_{-\infty}^{\infty} \sqrt{\frac{2}{a}} \exp\left(-u^2\right)\, du = \frac{\sqrt{2\pi}}{\sqrt{a}}.$$

This completes the proof. □

Derivation 8.25. Consider the differential equation:

$$f'(x) + axf(x) = 0$$

Apply the Fourier transform to both sides and use Theorem 8.11 to see:

$$\mathcal{F}\left[f'(x) + axf(x)\right] = (i\omega)\hat{f}(\omega) + ai\hat{f}'(\omega) = 0$$

Simplifying (divide by ai) we see that the Fourier transform $\hat{f}(\omega)$ must satisfy:

$$\hat{f}'(\omega) + \frac{1}{a}\omega\hat{f}(\omega) = 0.$$

Therefore, the solution from Lemma 8.21 is:

$$\hat{f}(\omega) = A\exp\left(-\frac{\omega^2}{2a}\right),$$

for some constant A to be determined. Evaluate the Fourier transform at $\omega = 0$. Note $e^{-i\omega x} = 1$ when $\omega = 0$. We have:

$$\hat{f}(0) = A = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx = \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{\sqrt{2\pi}}{\sqrt{a}}\right) = \frac{1}{\sqrt{a}}.$$

Thus:

$$\hat{f}(\omega) = \frac{1}{\sqrt{a}}\exp\left(-\frac{\omega^2}{2a}\right).$$

We have proved a theorem.

THEOREM 8.26. *Assume $a > 0$, then:*

$$\mathcal{F}\left[\exp\left(-\frac{ax^2}{2}\right)\right] = \frac{1}{\sqrt{a}}\exp\left(-\frac{\omega^2}{2a}\right).$$

5. Solving PDE's with the Fourier Transform

Remark 8.27. If we have $u(x,t)$ we can take the Fourier transform in either variable to obtain a new function in ω. For example we might take the Fourier transform with respect to x to obtain $\hat{u}(\omega, t)$. Given a PDE, we will generally use the variable(s) in which we have something like an initial condition or a functional boundary condition.

Derivation 8.28. Consider the heat equation on $\mathbb{R}$:

$$u_t = ku_{xx}$$
$$u(x,0) = f(x)$$

Take the Fourier transform on the variable x on both sides. Both the derivative and Fourier transform are linear so applying Corollary 8.12 to obtain:

$$\partial_t\hat{u}(\omega,t) = -k\omega^2\hat{u}(\omega,t).$$

This is a simple ODE in terms of t. We can solve it as:

$$\hat{u}(\omega, t) = A_0(\omega) \exp\left(-k\omega^2 t\right).$$

Here $A_0(\omega)$ is an unknown function of ω to be identified from the initial condition. (Remember in PDE's we have constants of integration become functions, just as they did when we derived D'Almbert's formula.) The initial condition us $\hat{u}(\omega, 0) = \hat{f}(\omega)$, the Fourier transform of f because $u(x, 0) = f(x)$. But then setting $t = 0$ we can see that

$$\hat{u}(\omega, t) = f(\omega) \exp\left(-k\omega^2 t\right). \tag{8.3}$$

We want:

$$-kt\omega^2 = -\frac{\omega^2}{2a},$$

to apply Lemma 8.24. This means that:

$$a = \frac{1}{2kt}.$$

Then from Eq. (8.3) we have:

$$\frac{1}{\sqrt{a}}\hat{u}(\omega, t) = \frac{1}{\sqrt{a}} f(\omega) \exp\left(-\frac{\omega^2}{2a}\right).$$

Now we can invert the Fourier transform (using Theorem 8.20):

$$\frac{1}{\sqrt{a}} u(x, t) = \exp\left(-\frac{ax^2}{2}\right) * f(x).$$

We can unwrap this (and replace a) to obtain:

$$u(x, t) = \frac{\sqrt{a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \exp\left(-\frac{a(x-s)^2}{2}\right) ds = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(s) \exp\left(-\frac{(x-s)^2}{4kt}\right) ds.$$

Remark 8.29. This expression should start to look familiar – see Proposition 1.23. In the next section, it will look even more familiar.

EXERCISE 49. Use the approach in Derivation 8.28 to solve the wave equation on $\mathbb{R}$:

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \\ &u(x, 0) = f(x) \\ &u_t(x, 0) = g(x) \end{aligned}$$

using a Fourier transform. You do not have to take the inverse Fourier transform. (It's messy, we'll discuss it in the next section.)

6. Generalized Functions & Fundamental Solutions

Remark 8.30. There are several ways to formally define the Dirac delta function. We'll use the limit of functions approach and hint at the linear functional method. It is also worth noting there are many ways to approach the limit of functions definition. We'll use the one from [Olv14] because it's pretty and uses a fact from Calculus 1. [Asm16] has a different approch.

Lemma 8.31. *The following equation holds:*

$$\int \frac{n}{\pi(1+n^2x^2)}\,dx = \frac{1}{\pi}\tan^{-1}(nx).$$

□

Definition 8.32. Let:

$$f_n(x) = \frac{n}{\pi(1+n^2x^2)}.$$

Then:

$$\delta(x) = \lim_{n\to\infty} f_n(x).$$

Remark 8.33. The functions $f_n(x)$ are illustrated in Fig. 8.3.

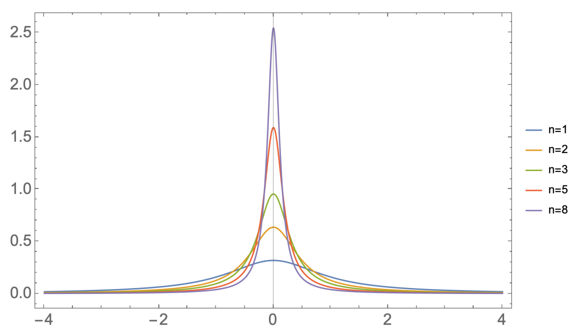

FIGURE 8.3. The limit of the functions $f_n(x)$ converges to what we call the Dirac Delta function.

Corollary 8.34. *The following equation holds:*

$$\int_{-\infty}^{\infty} \frac{n}{\pi(1+n^2x^2)}\,dx = 1.$$

PROOF. We are assuming n is a positive integer. We know:

$$\lim_{x\to\infty} \frac{1}{\pi}\tan^{-1}(nx) = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

$$\lim_{x\to-\infty} \frac{1}{\pi}\tan^{-1}(nx) = -\frac{1}{\pi}\frac{\pi}{2} = -\frac{1}{2}$$

The result follows immediately. □

Remark 8.35. Given Corollary 8.34 we would *hope* that:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty}\left[\lim_{n\to\infty} f_n(x)\right]\,dx = \int_{-\infty}^{\infty}\delta(x)\,dx = 1.$$

As we've discussed in the past, showing that the limit can be passed through the integral requires showing convergence properties that are well outside the scope of this course. More to the point, with ordinary Riemann (or even Lebesgue) integration, this equality is **incorrect**. To make this sensible mathematically requires an entirely different course on analysis. We are just going to accept it as true for expediency. More to the point we're actually going to accept the following theorem.

THEOREM 8.36. *Suppose $\xi \in [a,b]$. Then:*

$$\int_a^b \delta(x-\xi)\,dx = 1.$$

On the other hand, if $\xi \notin [a,b]$, then:

$$\int_a^b \delta(x-\xi)\,dx = 0.$$

□

Remark 8.37. Assume $g(x)$ is a function defined on $[a,b]$ with $\xi \in (a,b)$. Another interesting property of $f_n(x)$ is:

$$\lim_{n\to\infty}\langle f_n(x-\xi), g(x)\rangle = \lim_{n\to\infty}\int_a^b f_n(x-\xi)g(x)\,dx = g(\xi).$$

We can therefore think of the following (inner product) functional as defining the Dirac Delta function:

$$\int_a^b \delta(x-\xi)g(x)\,dx = g(\xi). \tag{8.4}$$

In this way, given an interval $[a,b]$ and a test point $\xi \in [a,b]$, we think of the Dirac Delta more as a functional:

$$L_\xi(g) = \langle \delta(x-\xi), g\rangle = g(\xi),$$

where $\delta(x-\xi)$ is the defining component that makes the functional work the way we want. This is a bit closer to the formal way to define this, but the important thing to take away is actually Eq. (8.4).

EXERCISE 50. Show that if $x \in (a,b)$, then:

$$\int_a^b \delta(x-s)g(s)\,dx = g(x). \tag{8.5}$$

[Hint: The Delta function is even so $\delta(x) = \delta(-x)$. Then $\delta(s-x) = \delta(x-s)$. Now apply Eq. (8.4).]

Derivation 8.38. Consider the function:

$$H(x) = \int_{-\infty}^{x} \delta(s)\,ds.$$

We see immediately from Theorem 8.36 and that this is the step function:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

This function is not defined at $x = 0$. This is the *Heaviside step function.* As a consequence of this result we say:

$$\frac{dH}{dx} = \delta(x).$$

Proposition 8.39. *The Fourier transform of the Dirac Delta function is:*

$$\mathcal{F}[\delta(x-\xi)] = \frac{e^{-i\xi\omega}}{\sqrt{2\pi}}$$

Proof. Evaluate:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x}\delta(x-s)\,dx = \frac{e^{-i\xi\omega}}{\sqrt{2\pi}}.$$

by Eq. (8.4). □

Remark 8.40. Plugging in $\xi = 0$, it's straightforward to see:

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}.$$

We use this fact later.

Exercise 51. The Inverse Fourier Transform of $\cos(\omega t)$ is:

$$\mathcal{F}^{-1}[\cos(\omega t)] = \sqrt{\frac{\pi}{2}}\delta(x-t) + \sqrt{\frac{\pi}{2}}\delta(t+x).$$

You can prove this using Proposition 8.39 and the definition of cosine in terms of the exponential function. Now consider the wave equation on $\mathbb{R}$:

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \\ &u(x,0) = f(x) \\ &u_t(x,0) = 0 \end{aligned}$$

Use the given inverse Fourier transform and Exercise 49 to arrive at D'Almbert's formula for the given PDE.

Definition 8.41 (Multi-Dimensional Dirac Delta)**.** The Dirac Delta function can be extended to multiple dimensions as the function:

$$\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\cdots\delta(x_n),$$

where $\mathbf{x} = (x_1, \dots, x_n)$. For example, $\delta(x,t) = \delta(x)\delta(t)$ and is a delta function that occurs at $t = 0$ and $x = 0$.

7. Fundamental Solutions and Green's Function

Remark 8.42. We begin this section by considering the heat equation on the line with the initial condition $u(x, 0) = \delta(x)$. This is an instance of the case where an infinite point of heat is positioned at the origin at the beginning of time and we determine the resulting heat distribution over time. The following proposition can be proved by applying the final equation of Derivation 8.28 and Eq. (8.4).

Proposition 8.43. *Consider the following heat equation on the real line:*

$$u_t = ku_{xx}$$
$$u(x, 0) = \delta(x)$$

Then:

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

Exercise 52. Fill in the details of the proof of Proposition Proposition 8.43.

Remark 8.44. You have seen the result from Proposition 8.43 before. This is the heat kernel we introduced in Eq. (1.10). We usually denote this $\Phi(x, t)$. At this point, Haberman [**Hab03**] stops, declaring that this is the fundamental solution. We can do a little better by taking a tour into Green's functions.

Definition 8.45 (Green's Function)**.** Consider a one-dimensional linear operator L with some given boundary conditions. We assume the operator works on the variable x; e.g., L is a derivative with respect to x. A *Green's function solution* is a function $G(x, s)$ satisfying the boundary conditions with the property that:

$$L[G(x, s)] = \delta(x - s).$$

Note: I'm using the preferred Physics definition of the Green's function. In math, it's defined as $L[G(x, s)] = \delta(s - x)$. As we know from Exercise 50, it makes no difference.

Remark 8.46. We first note that not every operator admits a Green's function solution. Second, if the operator is translation invariant (e.g., has constant coefficients, unlike a general SLEP), then the Green's function is a univariate function $G(z)$ and:

$$G(x, s) = G(x - s).$$

Derivation 8.47. Assume an operator L in one dimension admits a Green's function. It's worth asking, why would anyone care? Consider the following non-homogenous (and therefore hard) problem:

$$L[w(x)] = f(x),$$

where $w(x)$ is an unknown function satisfying some boundary conditions on an interval $[a, b]$, with $0 \in (a, b)$. (If zero is not in the interval, we can always shift the interval to make this true.) Suppose $G(x, s)$ is a Green's function. Then:

$$L[G(x, s)] = \delta(x - s).$$

Now consider the integral on $[a, b]$:

$$\int_a^b G(x, s) f(s)\, ds.$$

Apply the operator L and assume that L commutes with the integral (a big assumption).

$$L\left[\int_a^b G(x,s)f(s)\,ds\right] = \int_a^b L[G(x,s)]f(s)\,ds = \int_a^b \delta(x-s)f(s)\,ds = f(x),$$

for $x \in [a,b]$. Thus the Green's function can be used to solve the non-homogenous problem by the integral operator:

$$w(x) = I[f(s)] = \int_a^b G(x,s)f(s)\,ds.$$

When $G(x,s) = G(x-s)$, then this is just a variation on convolution:

$$w(x) = \int_a^b G(x-s)f(s)\,ds.$$

8. Laplace Transform and a Green's Function Example

Remark 8.48. In this section, our goal is to construct a useful example of a Green's function. In order to do this as quickly as possible, we're going to use the Laplace transform. I will state (but not prove) a few results that should have been covered in an introductory course in Ordinary Differential Equations. If this were a proper textbook, this would be in its own chapter and would be developed more completely.

Definition 8.49. Given a function $f(t)$, the *Laplace Transform* is:

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Remark 8.50. As an integral transform, the Laplace transform is linear.

Proposition 8.51. *The following Laplace transforms hold:*

$$\mathcal{L}\{\delta(t-\xi)\}(s) = e^{-\xi s}, \tag{8.6}$$

$$\mathcal{L}\{e^{kt}\}(s) = \frac{1}{s-k}, \tag{8.7}$$

$$\mathcal{L}\{H(t-\xi)f(t-\xi)\}(s) = e^{-\xi s}F(s), \tag{8.8}$$

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0). \tag{8.9}$$

□

Derivation 8.52. Consider the following operator:

$$L(w) = \frac{dw}{dt} - kw = 0,$$

where $k \in \mathbb{R}$ is a constant. Assume the initial condition $w(0) = 0$. Then the Green's function problem is to solve the problem:

$$\frac{dw}{dt} = kw + \delta(t-\xi)$$
$$w(0) = 0.$$

For simplicity, we assume $\xi \geq 0$. Notice we're replacing s with ξ in the Green's function because we now have an s in the Laplace transform. To solve this problem, apply the Laplace transform to both sides:

$$\mathcal{L}\left\{\frac{dw}{dt}\right\} = \mathcal{L}\left\{kw + \delta(t-\xi)\right\}.$$

Using Proposition 8.51, we have:

$$sW(s) - w(0) = kW(s) + e^{-\xi s}.$$

We know $w(0) = 0$ and we can solve for $W(s)$ to obtain:

$$W(s) = \frac{e^{-\xi s}}{s-k} = e^{-\xi s}\frac{1}{s-k}.$$

Now we can back infer that:

$$w(t) = H(t-\xi)e^{k(t-\xi)} = \begin{cases} e^{k(t-\xi)} & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

For consistency with the initial condition, when $\xi = 0$ we write:

$$w(t) = H(t)e^{kt} = \begin{cases} e^{kt} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Thus we have deduced that:

$$G(t,\xi) = \begin{cases} e^{k(t-\xi)} & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

We can conclude that given the problem:

$$\begin{aligned} &\frac{dw}{dt} - kw = f(t) \\ &w(0) = 0, \end{aligned}$$

we have:

$$w(t) = \int_0^t G(t,\xi)f(\xi)\,d\xi = e^{kt}\int_0^t e^{-k\xi}f(\xi)\,d\xi.$$

9. The Fundamental Solution of the Heat Equation

Derivation 8.53. Consider the following Green's Function like problem for the heat equation:

$$\begin{aligned} &u_t = ku_{xx} + \delta(t-\xi)\delta(x) \\ &u(x,0) = 0 \end{aligned}$$

Like the ordinary heat equation with initial condition $u(x,0) = f(x)$, this models an infinite heat impulse occurring at time $x = 0$ and $t = \xi$. (We will set $\xi = 0$ in a moment.)

Just as in Derivation 8.28, we take the Fourier transform in terms of x to obtain:

$$\partial_t\hat{u}(\omega,t) = -k\omega^2\hat{u}(\omega,t) + \frac{1}{\sqrt{2\pi}}\delta(t-\xi),$$

where:

$$\mathcal{F}[\delta(t-\xi)\delta(x)] = \frac{1}{\sqrt{2\pi}}\delta(t-\xi),$$

from Proposition 8.39 and the fact that functions of t are treated as constants when we take the Fourier transform in terms of x.

Now take the Laplace transform in terms of t as in Derivation 8.52 to obtain:

$$s\hat{U}(\omega,s) - \hat{u}(\omega,0) = -k\omega^2\hat{U}(\omega,s) + \frac{1}{\sqrt{2\pi}}e^{-\xi s}.$$

We know:

$$\hat{u}(\omega,0) = 0,$$

because $u(x,0) = 0$. Therefore, we deduce:

$$\hat{U}(\omega,s) = \frac{1}{\sqrt{2\pi}}e^{-\xi s}\frac{1}{s+k\omega^2}.$$

Take the inverse Laplace transform first to see:

$$\hat{u}(\omega,t) = \begin{cases} \frac{1}{\sqrt{2\pi}}e^{-k\omega^2(t-\xi)} & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

With respect to the inverse Fourier transform, $t-\xi$ is a constant. As in Derivation 8.28 let:

$$a = \frac{1}{2k(t-\xi)}$$

We have:

$$\frac{1}{\sqrt{a}}\hat{u}(\omega,t) = \begin{cases} \frac{1}{\sqrt{a}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\omega^2}{2a}\right) & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

Now invert the Fourier transform to see:

$$\frac{1}{\sqrt{a}}u(x,t) = \begin{cases} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{ax^2}{2}\right) & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

Now replace a and simplify to see:

$$u(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi k(t-\xi)}}\exp\left(-\frac{x^2}{4k(t-\xi)}\right) & \text{if } t > \xi \\ 0 & \text{if } t < \xi. \end{cases}$$

Let $\xi = 0$ and we obtain the *fundamental solution* (which is really a Green's function solution):

$$\Phi(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi kt}}\exp\left(-\frac{x^2}{4kt}\right) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

That means we can use this to solve the non-homogeneous problem on (with zero initial condition) on $\mathbb{R}$:

$$(\partial_t - \partial_{xx})\,u = f(x,t),$$

by computing the (2D) convolution:

$$\Phi * f = \int_0^\infty \int_{-\infty}^\infty \Phi(x - s, t - \tau) f(s, \tau)\, ds\, d\tau.$$

Remark 8.54. We note that the fundamental solution is a *weak* solution since the derivative clearly does not work over the jump discontinuity. However, we've gotten use to this looseness in our notion of differentiable by now. See [**Eva15**] for the details.

Exercise 53. Argue using super-position that a solution to the generic forced heat equation on $\mathbb{R}$:

$$\begin{aligned} &u_t = k u_{xx} + f(x,t) \\ &u(x,0) = h(x) \end{aligned}$$

is given by:

$$u(x,t) = \int_0^\infty \int_{-\infty}^\infty \Phi(x - s, t - \tau) f(s, \tau)\, ds\, d\tau + \int_{-\infty}^\infty \Phi(x - s, t) h(s)\, ds.$$

[Hint: Go to Chapter 7 and think of how we solved the Poisson problem with boundary conditions. Setup a heat equation with no forcing and an initial condition and a second heat equation with initial condition 0 and forcing. Argue you can solve them independently and add them.]

Remark 8.55. This method of analysis can be extended to more complex boundary conditions or initial conditions. We will not do so here, but it is covered in the referenced texts (e.g.) [**Olv14, Asm16**].

10. Gravity in Two and Three Dimensions

Remark 8.56. We return (one more time) to Laplace's equation and seek a circular or spherically symmetric Green's function solution in two and three dimensions. As a by-product, we derive Newton's laws of gravity (up to a constant).

Remark 8.57. In three dimensions, the gravitational potential $u(\mathbf{x})$ for a mass-density $\rho(\mathbf{x})$ must satisfy the Poisson equation:

$$\Delta u = 4\pi G \rho(\mathbf{x}).$$

Solving this Poisson equation – which is really the non-homogeneous Laplace equation – can be accomplished with a Green's function approach. The convolution is a little more complex, but the principle is the same. We first analyze this problem in two dimensions.

Lemma 8.58. *Let $\mathbf{r} \in \mathbb{R}^2$ be a variable position along the circle S parameterized by $[r\cos(\theta), r\sin(\theta)]$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Then:*

$$\oint_S f(\mathbf{r})\, ds = \int_0^{2\pi} f(\mathbf{r}) r\, d\theta.$$

Proof. Recall the formula for a line integral is:

$$\oint_S f(\mathbf{r})\, ds = \int_0^{2\pi} f(\mathbf{r}) |\mathbf{r}'(\theta)|\, d\theta.$$

We compute:

$$\mathbf{r}' = \begin{bmatrix} -r\sin(\theta) \\ r\cos(\theta). \end{bmatrix}$$

Therefore:

$$|\mathbf{r}'(\theta)| = \sqrt{r^2\cos^2(\theta) + r^2\sin^2(\theta)} = r.$$

The result follows by substitution. ☐

Derivation 8.59. Consider the 2D Poisson equation:

$$-\Delta u = \delta(\mathbf{x}).$$

Ignoring constants, this models a (Newtonian) gravitational potential with an infinitely dense point-mass at the origin. We use some physical intuition and realize we seek a circularly symmetric solution u, so we use polar coordinates. Recall the polar Laplacian is:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

For circular symmetry, the derivatives in θ are all zero and the PDE reverts to an ODE:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \delta(r).$$

We can bring the Dirac delta requirement in by replacing it with a boundary condition, just as we obtained the fundamental solution for the heat equation with an initial condition. We must solve:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0$$
$$\lim_{r\to 0} u(r) = +\infty.$$

We also require these solutions to be circularly symmetric. This problem is equivalent to the Cauchy-Euler ODE in Corollary 5.16 when $n = 0$ (multiply by r^2 to see this). In fact, we encountered this ODE the last time we analyzed the Laplace equation. We know from Corollary 5.16 that the solution in this case is:

$$u(r) = c_1 + c_2 \log(r).$$

The challenge is to determine the coefficients. From physics, we know that since $u(r)$ is a potential function, any constant can be added to it and it's still a potential function. (Potential functions are only useful when you differentiate them.) So we are free to set $c_1 = 0$. (Olver [**Olv14**] justifies this by saying that c_1 has zero derivative so cannot contribute to the $\delta(r)$ function.) We now obtain c_2 using Green's Second Identity.

$$\int_\Omega u\,\Delta v\,d\Omega - \int_\Omega v\,\Delta u\,d\Omega = \int_{\partial\Omega} [u\,\nabla v - v\,\nabla u]\cdot\mathbf{n}\,dS.$$

Let Ω be a small disk of radius r around the origin and $S = \partial\Omega$. Set $v = 1$ to see:

$$\int_\Omega \Delta u\,d\Omega = \int_S \nabla u\cdot\mathbf{n}\,dS.$$

Now we plug in by arguing that $\Delta u = \delta(r)$. We see first that:

$$\int_\Omega \Delta u\,d\Omega = \int_\Omega -\delta(r)\,d\Omega = -1,$$

using a generalization of Theorem 8.36. Now we compute:

$$\nabla u = \frac{c_2}{r},$$

because we are only differentiating in terms of r (remember the circular symmetry). Now S is just a circle so the surface integral is really a line integral. Applying Lemma 8.58, we can convert this to an ordinary integral as:

$$\int_S \nabla u \cdot \mathbf{n}\, dS = \int_0^{2\pi} \frac{c_2}{r} r\, d\theta = 2\pi c_2.$$

Therefore, we conclude that:

$$-1 = 2\pi c_2,$$

or:

$$c_2 = -\frac{1}{2\pi}.$$

The fundamental solution is then:

$$\Phi(r) = \frac{-\log(r)}{2\pi}.$$

We can convert this back to Euclidean coordinates and see:

$$\Phi(x, y) = \frac{-\log(\sqrt{x^2 + y^2})}{2\pi}.$$

Now compare this to Exercise 9. This is a potential function for gravity in a two dimensional world and suggests that the correct form of Gauss's law for gravity should be:

$$\nabla \mathbf{g} = -2\pi G\rho(\mathbf{x}).$$

The 2π will cancel the denominator of the fundamental solution of the Laplace equation in 2D. We will return to this after we compute a similar result in three-dimensional space.

Remark 8.60. In order to consider the Laplace and Poisson equation in 3D space, we need a lemma and a definition.

Definition 8.61 (Laplacian in Spherical Coordinates)**.** The *Laplacian in spherical* coordinates is:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial^2}{\partial \varphi^2}.$$

Here coordinates are given in (r, θ, φ). The angle θ is the polar angle and the angle φ is the azimuthal angle.

Lemma 8.62. *Let $\mathbf{r} \in \mathbb{R}^3$ be a variable position along the sphere S parameterized by the vector valued function:*

$$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} r\cos(\theta)\sin(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\varphi) \end{bmatrix}$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function. Then:

$$\oiint_S f(\mathbf{r})\, ds = \int_0^{2\pi}\int_0^{\pi} f(\mathbf{r}) r^2 \sin\varphi\, d\varphi\, d\theta.$$

PROOF. Apply the formula

$$\oiint_S f(\mathbf{r})\,ds = \int_0^{2\pi}\int_0^{\pi} f(\mathbf{r}) \left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right\| d\varphi, d\theta.$$

Computing the norm of the cross-product yields:

$$\left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right\| = r^2 \sin(\varphi).$$

The result follows immediately. □

Remark 8.63. We now proceed with a derivation very similar to Derivation 8.59 but in 3D. We will proceed more quickly than before.

Derivation 8.64. Consider the 3D Poisson equation:

$$-\Delta u = \delta(\mathbf{x}),$$

which models the gravitational potential of an infinite density point-mass at the origin in 3D (again ignoring constants). We assume spherical symmetry and therefore the Laplace equation becomes a one-dimensional ODE

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{\partial r}\right) = \delta(r).$$

We can use the boundary condition trick and simplify the ODE to obtain:

$$\frac{2u'(r)}{r} + u''(r) = 0$$
$$\lim_{r\to 0} u(r) = +\infty.$$

Let $v(r) = u'(r)$. Then the ODE becomes:

$$\frac{2}{r}v(r) + v'(r) = 0.$$

This can be solved with separation of variables to see that:

$$v(r) = \frac{c_1}{r^2}.$$

Integrating we immediately see that:

$$u(r) = \frac{c_1}{r} + c_2.$$

As before, let $c_2 = 0$, this is a potential function and we are free to ignore it. We must find the value of c_1. We proceed as before we use Green's Second Identity. As before let Ω be a small ball of radius r around the origin and $S = \partial\Omega$. Set $v = 1$ and proceed as before to obtain:

$$\int_\Omega \Delta u\, d\Omega = \int_S \nabla u \cdot \mathbf{n}\, dS.$$

Now the integral on the left is a volume integral and the integral on the right is a surface integral. Just as before:

$$\int_\Omega \Delta u\, d\Omega = \int_\Omega -\delta(r)\, d\Omega = -1,$$

Now we compute:

$$\nabla u = -\frac{c_2}{r^2}.$$

The surface S is just the surface of a sphere. Applying Lemma 8.62 we have:

$$\int_S \nabla u \cdot \mathbf{n}\, dS = \int_0^{2\pi}\int_0^{\pi} -\frac{c_2}{r^2} r^2 \sin(\varphi)\, d\varphi\, d\theta = -4\pi c_2.$$

Thus we conclude:

$$c_2 = \frac{1}{4\pi}.$$

The fundamental solution is then:

$$\Phi(r) = \frac{1}{4\pi r}.$$

We can convert this back to Euclidean coordinates and see:

$$\Phi(x, y, z) = \frac{1}{4\pi(\sqrt{x^2 + y^2 + z^2})}.$$

This explains Gauss' law of gravitation in 3D, which is:

$$\nabla \mathbf{g} = -4\pi G \rho(\mathbf{x}).$$

The 4π cancels the same value that appears in the denominator of the fundamental solution and also is a constant that comes out of three dimensional space just as 2π arises out of two dimensional space.

Derivation 8.65 (Inverse Square? Law)**.** We now turn our attention to the force of gravity in 2D and 3D. Given a potential function u for a force, we know the force is proportional to:

$$\mathbf{F} = \nabla u.$$

We have identified the fundamental solutions of the Laplace equation. We must simply adjust them with constants to model gravity. In three-space, the potential function for the Dirac delta mass distribution is:

$$u = -4\pi G \Phi(r).$$

Here:

$$\Phi(r) = \frac{1}{4\pi r}.$$

Setting $G = 1$, we have the force of gravity in three dimensions is:

$$F = -4\pi \nabla \Phi(r) = -\frac{4\pi}{4\pi r^2}\hat{\mathbf{r}} = -\frac{1}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is a unit vector pointing from the origin to the point (x, y, z). (The negative means gravity pulls you toward the point-mass). We have proved that in three-space, Newtonian gravity follows an inverse square law. That is, rather than starting from this assumption, we end with it after analyzing gravity with the Laplace equation.

In two dimensions, things are quite different. Recall:

$$\Phi(r) = \frac{-\log(r)}{2\pi}.$$

We have the potential function:

$$u = 2\pi G\Phi(r).$$

Again set $G = 1$ and observe that:

$$F = 2\pi\nabla\Phi(r) = -\frac{2\pi}{2\pi r}\hat{\mathbf{r}} = -\frac{1}{r}\hat{\mathbf{r}}.$$

Here $\hat{\mathbf{r}}$ is a vector pointing from the origin to (x, y). In a two dimensional universe, gravity follows an inverse-linear law!

Remark 8.66. It is worth noting that Coulomb's Law for electric charges can be derived using the same machinery (in fact Gauss' law is essentially identical). In this respect, gravity and electromagnetism are identical to each other. But what an unusual place Flatland must be! Gravity and electric forces vary with inverse-linear laws. Information travels *inside waves*, making it noisy and blurry! How exciting it is that we can sit in our comfortable three-dimensional universe and know what it's like in Flatland without having to go there ourselves. Thanks PDE's.

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